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Third Paper on Tensor Analysis.*

By G. Y. Rainich.

This paper deals with some questions of the integral calculus of tensor fields using the method and notions introduced in the foregoing papers.

Before we state, by way of introduction, what questions are treated here, it will be well to point out a question which will not be discussed in the body of the paper. The differentiation of a tensor field of rank k leads to a tensor field of rank $k+1$ (First paper §§ 4 and 9; second paper § 18). It is natural, given a tensor field of rank $k+1$, to ask whether it may be considered as a differential of a field of rank k and, if so, to try to find this tensor field. In the case $k=0$ the given field would be a field of rank *one*, i. e. a vector field, and the questions whether this field is a gradient and, if so, how to find the scalar field of which it is the gradient, are answered in a way entirely analogous to that followed in a Euclidean space. If, however, $k>0$, the situation becomes radically different: let us consider the case $k=1$, for instance, using the customary coordinate notations; if ϕ_i is a tensor field of rank one, if $\phi_{i,j}$ is its first and $\phi_{i,jk}$ its second differential, it is well known that

$$\phi_{i,jk} - \phi_{i,kj} = R^{\rho}{}_{i;jk}\,\phi_{\rho},$$

where $R^{t}{}_{i;jk}$ is the Riemann tensor of the space. If therefore a tensor field ψ_{ij} of rank two is given we see that, if it is the differential of a tensor field ϕ_i it must satisfy the relation

$$\psi_{ij,k} - \psi_{ik,j} = R^{\rho}{}_{i,jk}\,\phi_{\rho},$$

which in a general space determines the primitive field completely. We have thus a true "integration by differentiation" and the primitive field is determined uniquely, *there are no constants of integration.* We shall not discuss in this paper these interesting questions, which, it seems, are also important from the physical point of view; we shall not deal here with the research of primitive functions, not with questions which correspond to the study of the indefinite integral but with the process of integration, with the generalization of the definite integral.

We shall follow the system adopted in the first two papers, beginning

* The two first papers appeared in this *Journal*, Vol. XLVI, No. 2, April, 1924, pp. 71-94 and Vol. XLVII, No. 1, January, 1925, pp. 11-24.

with the study of the simpler case of a surface in the ordinary space; the indicator, as it ought to be, shows the way; it shows that integration leads away from the surface into the space; we then take up the general case of a tensor field on the surface, show what integration becomes in this case and deduce the integrability conditions or the conditions for the independence of the integral of the path; besides a differential condition which is the exact counterpart of the usual condition we have here additional algebraic conditions. Before entering upon the discussion of the generalization of these things for higher spaces we show how the Riemann tensor can be introduced in a new way which in many respects resembles the introduction of the indicator in the case of a surface. The idea of tensor area seems to prove useful in this connection as well as elsewhere. After we introduce surface integration and volume integration for higher spaces and discuss the integrability conditions following the way adopted in the case of the surface we treat the Riemann tensor and the contracted Riemann tensor as examples. In the last section we indicate a method of obtaining integral expressions without leaving the curved space.*

§ 21. *Line Integrals on a Surface.*

We remember that the (negative) indicator was obtained in our first paper (7.5) as

$$\lim \frac{b-a}{\lambda} \quad \text{with} \quad \frac{B-A}{\lambda} \to h,$$

where b and a were unit normal vectors at the points B and A respectively. We may consider the totality of unit normal vectors as a vector field (which, however, does not *belong* to the surface). If we introduce the notation $a=f(A)$, $b=f(B)$ we have

$$-s(h)=\lim \frac{f(B)-f(A)}{\lambda},$$

so that the negative indicator appears as the differential of this vector field (compare 4.1). We may also say that the indicator gives the *infinitesimal* change of the unit normal vector which corresponds to the infinitesimal displacement h on the surface. If we write dA for h, we have

$$da=-s(dA).$$

* The contents of this paper was briefly outlined in papers read before the American Mathematical Society on December 28, 1924 and February 27, 1925.

It is natural to consider the *finite* change of the unit normal vector $q - p$ as the integral of $-s(dA)$,

$$q - p = -\int_P^Q s(dA).$$

This suggests going outside of the surface if we want to consider integrals.

Following this suggestion we shall find an operation of integration of which this transition from $s(dA)$ to $q - p$ is a special case. One way to do this is to consider the integral as the limit of a sum. It is, however, not necessary to introduce this process explicitly, as it is possible to reduce the case which we have to consider to things already known. The hint which we have, viz., that we have to take into account the outside space, will help us to make this reduction.

The situation is this. If we have a vector field, i. e. a vector v_A at each point A, we form the integral

$$\int v_A . dA$$

where $v_A . dA$ is the scalar product of the vectors v_A and dA; this can be considered as the limit of the sum

$$v_A . (B - A) + v_B . (C - B) + v_C . (D - C) + \cdots,$$

where $A, B, C, D, \cdots$ are points along a curve, and in coordinates has the familiar form

$$\int v_x dx + v_y dy + v_z dz.$$

Instead of considering a vector v we may consider (property 1, § 3, First Paper) the scalar linear function—tensor of rank one—$\phi(x) = v.x$; our integral takes then the form

$$\int \phi(dA).$$

If we have, say, a tensor of rank two, $\phi(x, y)$ and our surface happens to be *a plane*, we may give to the second argument y a fixed value and form, without any difficulty, the integral

$$\int \phi(dA, y),$$

(for every given path of integration) a function of y, to be precise a tensor of rank one whose argument is y, instead of a number which we obtained before—and the same can be applied to tensors of higher ranks. But, if we have a curved surface, we encounter the same difficulty as in the case of differentiation (§ 9): we cannot give to y a fixed value. We therefore resort

to the *following procedure* which is of fundamental importance for all that follows:

We take a fixed, but arbitrary vector c of the space which contains our surface; we consider in every point A of the surface a vector c_A which is the projection of c on that bundle (on the tangent plane at A) and in forming the integral, we give to y at each point A the value c_A, viz.,

$$\int \phi(dA, c_A).$$

We obtain thus (for every path of integration) a number, say ψ. Now we make c variable (this corresponds to the restoration of variability to y in the case of the plane). The number ψ becomes a function of the vector c, and, as we easily see a linear function (using the fact that the projection of the sum of two vectors is the sum of their projections and the definition of a linear function in § 3, First paper). We thus obtain as a result of integration of a tensor field $\phi(x, y)$ of rank two (for a given path) a tensor of rank *one* but this tensor is *a tensor of the three dimensional containing space*. We may, again, consider instead of the tensor $\psi(c)$ the vector u defined by the relation $\psi(c) = u.c$ (§ 3) and instead of the tensor of the second rank $\phi(x, y)$ we may consider, at each point of the surface, the equivalent linear vector function $g(x)$ which is given by $g(x).y = \phi(x, y)$ and say that the result of integration of a linear vector function, which belongs to a surface, along a path on this surface is a vector of the containing space.

The indicator being a linear vector function, and the difference between the unit normal vectors at the ends of a path being a vector of the containing space, we see easily that we have actually found the generalization which we set out to find.

The further generalization for fields of higher ranks is obvious. In order not to complicate the writing we shall only mention the case of rank three; we take in this case *two* arbitrary vectors of the containing space c and q, consider their projections c_A and q_A, form as before the tensor field of rank one $\phi(dA, c_A, q_A)$, integrate this along a line Γ and obtain a tensor of the containing space $\psi(c, q)$ which has to be considered as the integral:

$$\int_\Gamma \phi(dA, c_A, q_A) = \psi(c, q).$$

If the given tensor field is not symmetric we may also form two other integrals

$$\int_\Gamma \phi(c, dA, q) \quad \text{and} \quad \int_\Gamma \phi(c, q, dA).$$

§ 22. *Integrability conditions.*

In the case of the indicator we know, from the geometrical meaning of the integral $\int_\Gamma s(dA)$, which gives the change of the unit normal vector along the curve Γ, that if two curves Γ_1 and Γ_2 have the same end points

$$\int_{\Gamma_1} s(dA) = \int_{\Gamma_2} s(dA),$$

provided the curves may be continuously transformed one into the other without leaving the surface, because each of the two integrals represents the difference between the normal unit vectors at the ends of the path. This suggests considering the general question of when, given an integral of a tensor field of rank two, it depends only on the end points of the path of integration and not on the path itself or when the integral taken over a closed reducible contour vanishes. In order that this be so $\int_\Gamma \phi(da, c_A)$ must be independent of the path for every value of c. If we put, for a given c, $\phi(x, c_A) = \omega(x)$ we know that the condition of independence is

$$\omega'(x, h) = \omega'(h, x) \tag{22.1}$$

where $\omega'(x, h)$ is the differential of the tensor field $\omega(x)$. (It would lead us too far if we desired to give here the vector deduction of this result which is well known in coordinate form). We have, therefore, to find the differential of the tensor field $\phi(x_A, c_A) = \omega(x_A)$, where c_A is the projection of a *constant* vector of the containing space. We know (First paper, § 9) that the differential of $\omega(x_A)$ or, to be more explicit, $\omega(A, x_A)$ is given by

$$\omega'(A, x_A, h_A) = \lim \frac{\omega(B, x_{A.B}) - \omega(A, x_A)}{\lambda},$$

$$\text{with } \frac{B - A}{\lambda} \to h_A$$

where $x_{A.B}$ is the projection of x_A on the bundle B. We therefore have

$$\omega'(A, x_A, h_A) = \lim \frac{\phi(B, x_{A.B}, c_B) - \phi(A, x_A, c_A)}{\lambda}. \tag{22.2}$$

This is *not* the differential of the tensor field of the second rank $\phi(A, x_A, y_A)$ for $y_A = c_A$. To have the differential we must have in the first term of the numerator $c_{A.B}$, the projection of c_A on the bundle B, instead of c_B, the projection of c on the bundle B. However, this coincides with the

differential $\phi'(A, x_A, c_A)$, if c coincides with c_A, i. e. if the vector c belongs (or is parallel) to the bundle A. In this case 22.1 takes the form

22.3 $$\phi'(x_A, c_A, h_A) = \phi'(h_A, c_A, x_A).$$

Since 22.1 must hold for all values of c, 22.3 must be satisfied, it is a *necessary* condition of integrability; in order to find sufficient conditions we will have to consider the case when c *does not* belong to the bundle A. Before we do this we notice that, applied to the indicator, equation 22.3 gives us the Codazzi equations of which we have thus found a derivation which seems to be novel.

Coming now to the general case we may present c as the sum of a vector y_A of the bundle A and of a vector which is normal to this bundle and which we may write ρa, a being the unit normal vector and ρ a number. We have thus

$$c = y_A + \rho a, \quad c_A = y_A, \quad c_B = y_{A.B} + \rho a_B;$$

22.2 takes the form

22.21 $$\omega'(A, x_A, h_A) = \lim \frac{\phi(B, x_{A.B}, y_{A.B}) - \phi(A, x_A, y_A) + \rho\phi(A, x_A, a_B)}{\lambda}$$

$$= \phi'(A, x_A, y_A, h_A) + \rho\phi(A, x_A, \lim a_B/\lambda)$$

We have to find now $\lim a_B/\lambda$. The vector a_B is the projection of the unit normal vector at A on the tangent plane at B and is easily shown to be $a - b(ab)$ using the reasoning employed in proving formula 9.2. We have now

$$\lim a_B/\lambda = \lim \frac{a - b(ab)}{\lambda} = \lim \frac{a - b}{\lambda}$$

$$+ \lim \frac{b[(a - b).a]}{\lambda} = s(h)$$

(compare the similar computation in § 12). We have thus

22.4 $$\omega'(A, x_A, h_A) = \phi'(A, x_A, y_A, h_A) + \rho\phi(A, x_A, s(h_A)).$$

Since 22.1 must hold for an arbitrary c we have to consider here y_A and ρ as arbitrary. Making $\rho = 0$ we obtain once more condition 22.3; making $y_A = 0$, $\rho = 1$ we obtain from 22.1

22.5 $$\phi(x, s(h)) = \phi(h, s(x))$$

where the letter A is dropped as argument and as index. This together with 22.3 which we write more conveniently as

22.6 $$\phi'(x, y, h) = \phi'(h, y, x),$$

gives a set of necessary and sufficient conditions of integrability.

If we have the integrand in the form of a vector function $f(x)$, the condition 22.5 becomes

22.7 $$f(x) . s(h) = f(h) . s(x)$$

which may also be written as

22.71 $$sf(x) . h = sf(h) . x$$

and, in the case when f is symmetric, in the even simpler form

22.72 $$sf(x) = fs(x) \quad \text{or} \quad sf = fs$$

which shows that the function f has to be commutative with the indicator s; condition 22.6 takes here the form

22.8 $$f'(x, h) = f'(h, x).$$

In the case of the indicator condition 22.7 is clearly satisfied, both sides becoming identically equal.

It may not be superfluous to insist on the fact that we have here, besides the usual differential condition of integrability, an additional algebraic condition involving the indicator. If both these conditions are satisfied for a tensor field of rank two which belongs to the surface there exists an ∞^1 of vector fields which are defined at all points of the surface but do not *belong* to the surface, because their vectors instead of being tangent stick out of it. All these fields may be obtained from one of them by the addition to all vectors of an arbitrary constant (space) vector.

If the integrand is a field of rank three $\phi(x, y, z)$ we have to put $c = y_A + \rho a$, $q = z_A + \sigma a$; we obtain instead of 22.21 and 22.4

$$\omega'(A, x_A, h_A) = \lim \frac{\phi(B, x_{A.B}, y_{A.B}, z_{A.B}) - \phi(A, x_A, y_A, z_A)}{\lambda} +$$

$$+ \lim \frac{\rho\phi(A, x_A, a_B, z_A) - \sigma\phi(A, x_A, y_A, a_B) + \rho\sigma\phi(A, x_A, a_B, a_B)}{\lambda}$$

$$= \phi'(A, x_A, y_A, z_A, h_A) + \rho\phi(A, x_A, s(h_A), z_A) + \sigma\phi(A, x_A, y_A, s(h_A))$$

the term containing a_B twice vanishing in the limit.

The integrability condition consists of three relations, one differential:

22.61 $$\phi'(x, y, z, h) = \phi'(h, y, z, x)$$

and two algebraic:

22.51 $$\phi(x, s(h), z) = \phi(h, s(x), z)$$

and

22.52 $$\phi(x, y, s(h)) = \phi(h, y, s(x))$$

§ 23. *Line Integrals in Higher Spaces.*

If, instead of a surface of the ordinary space, we consider an n-S. immersed in an N.-E.S., line integrals of tensor fields can be introduced in exactly the same way. In deducing the integrability condition a change has to be made, however. The vector c which belongs to the containing space must be presented as $y_A + a$, where a is an arbitrary normal vector; we cannot put it in the form ρa there being no definite unit normal vector when $r > 1$. Formula 22.21 takes the form

23.1 $$\omega'(A, x_A, h_A) = \phi'(A, x_A, y_A, h_A) + \phi(A, x_A, \lim a_B/\lambda)$$

where a is an arbitrary normal vector. We have now to find $\lim a_B/\lambda$ in this case.

We shall use this occasion and take up again the question of introducing the indicator dealt with in § 17 of the second paper and then in the added remark at the end of it. We consider at each point B of the neighborhood of A, B approaching A in such a way that $(B - A)/\lambda \to k_A$, a tangential vector h_B and a normal vector b, the values of these vectors for A being h_A and a respectively. We have then

23.2 $$h_B . b = 0 \qquad h_A . a = 0$$

We suppose that $\dfrac{h_B - h_A}{\lambda}$ and $\dfrac{b - a}{\lambda}$ approach finite limits as λ approaches zero so that

$$\lim \frac{(h_B - h_A)(b - a)}{\lambda} = 0.$$

This gives

$$\lim \frac{h_B . b}{\lambda} - \lim \frac{h_A . b}{\lambda} = \lim \frac{h_B . a}{\lambda} - \lim \frac{h_A . a}{\lambda}$$

or in view of 23.2

23.3 $$-\lim \frac{h_A . b}{\lambda} = \lim \frac{h_B . a}{\lambda} .$$

Since the left hand side of the equality does not contain h_B and the right hand side does not contain b we conclude that the common value of both sides

does not depend on either. This common value is, therefore, only a function of the vectors h_A, a and, of course, of k_A. Let us denote this function which, obviously, is a linear function of each of the three vectors, by $\psi(h_A, k_A, a)$, so that we have

$$23.4 \qquad \psi(h_A, k_A, a) = -\lim \frac{h_A.b}{\lambda} = \lim \frac{h_B.a}{\lambda} .$$

Using property 3 of § 3 (First paper) we may write ψ also in one of the following forms

$$23.5 \qquad \psi(h_A, k_A, a) = h_A \mathfrak{F}(a, k_A) = a.s(h_A, k_A)$$

where $\mathfrak{F}(a, k_A)$ is the tangential component of $\lim \frac{a-b}{\lambda}$ and $s(h_A, k_A)$ the normal component of $\frac{h_B - h_A}{\lambda}$. The functions $\mathfrak{F}$ and s have been introduced in the second paper ($\mathfrak{F}$ with the opposite sign). Here $\mathfrak{F}$ (a, k_A) is a linear vector function one of whose arguments is a tangential and the other a normal vector, while its value is a tangential vector; both arguments of s are tangential vectors and its value is normal; ψ is a scalar trilinear function with two tangential and one normal argument. The functions ψ, $\mathfrak{F}$, s are only different forms of the same thing. We shall refer to all three of them indifferently as to the indicatrix (reserving the name indicator for the function $s(x)$ which exists only in the case $N - n = 1$).

We return now to our question: What is $\lim a_B/\lambda$? It is clear, first of all, that it is a tangential vector. Denoting by a_b the normal component of a at the point B we may write

$$a_B = a - a_b$$

and

$$\lim \frac{a_B.h_A}{\lambda} = \lim \frac{(a-a_b).h_A}{\lambda} = -\lim \frac{a_b.h_A}{\lambda} .$$

Making now $b = a_b$ in the formulas 23.4 and 23.5 we may write

$$h_A.\mathfrak{F}(a, k_A) = \lim \frac{a_b.h_A}{\lambda} = -\lim \frac{a_B h_A}{\lambda} = -h_A.\lim \frac{a_B}{\lambda}$$

and, since $\lim a_B/\lambda$ is tangential, as remarked,

$$23.6 \qquad \lim a_B/\lambda = -\mathfrak{F}(a, k_A).$$

There would be some advantage in taking this as the definition of $\mathfrak{F}$. Relation 23.1 becomes now

$$\omega'(x, h) = \phi'(x, y, h) - \phi(x, \mathfrak{F}(a, h)).$$

Since 22.1 must hold for all values of $c = y + a$ it must hold for $a = 0$ and $y = 0$ so that we have

$$\phi'(x, y, h) = \phi'(h, y, x)$$

and

23.7 $$\phi(x, \mathfrak{F}(a, h)) = \phi(h, \mathfrak{F}(a, x))$$

and this must hold for every normal vector a. The formula 23.7 gives us the required algebraic integrability condition in the case now under consideration; relation 22.5 is, of course, a special case of this, $\mathfrak{F}(a, h)$ becoming $s(h)$ in the case when there exists a unique unit normal vector.

Using the linear vector function $f(x)$ instead of the tensor of the second rank ϕ we may write our relation as

$$f(x) . \mathfrak{F}(h, a) = f(h) . \mathfrak{F}(x, a)$$

or, with the aid of 23.5,

$$s[f(x), h] . a = s[f(h) . x] . a.$$

Since this must hold for an arbitrary a,

$$s[f(x), h] = s[f(h), x].$$

The generalization of 23.7 for the case of a tensor of higher rank is immediate. For rank three we have, for instance, outside of the differential condition the two algebraic conditions

23.8 $$\phi[x, \mathfrak{F}(a, h), z] = \phi[h, \mathfrak{F}(a, x), z]$$

and

23.81 $$\phi[x, y, \mathfrak{F}(a, h)] = \phi[h, y, \mathfrak{F}(a, x)] .$$

§ 24. *The Tensor Area.*

In higher spaces the Riemann tensor plays in some respects the same rôle that the indicator plays in the case of an ordinary surface (and, more generally, in the case $N - n = 1$). We considered the indicator as giving the infinitesimal change of the unit normal vector which corresponds to an infinitesimal displacement along a curve on the surface, and its integral gave us the finite change of the unit normal vector which corresponds to a finite displacement along a curve. In the case which we are about to consider now the rôle played by the curve will be played by a surface belonging to the n-S. considered; instead of the change of the normal unit vector between the two ends of a finite curve (which constitute the frontier of that curve) we shall

consider the change in the orientation of normal spaces which corresponds to the frontier of a finite piece of surface, i. e. to a contour, and we shall see that in the case of an infinitesimal contour the corresponding infinitesimal change of the normal spaces is connected with the Riemann tensor. Before we come to this, however, we have to introduce a new notion which has the same relation to the frontier of a piece of surface as the vector chord, i. e. the vector joining the ends of an arc, has to these two points which constitute the frontier of this arc. We consider here an N.-E.S.

The vector which joins the points A and B we agreed to represent by $B-A$; we know also that the consideration of a vector is equivalent to the consideration of its product with a variable vector—a linear scalar function (First paper, 3.4). In our case this tensor is $(B-A).x$, but it is more convenient to write it in the more complicated form

$$a(x) = \begin{vmatrix} 1 & (A-P).x \\ 1 & (B-P).x \end{vmatrix},$$

where P is an arbitrary point on which the expression obviously does not depend.

It is suggested, if we are given three points A, B, C, to consider the expression

$$24.1 \qquad \beta(A,B,C;\,x,y) = \tfrac{1}{2} \begin{vmatrix} 1 & (A-P).x & (A-P).y \\ 1 & (B-P).x & (B-P).y \\ 1 & (C-P).x & (C-P).y \end{vmatrix}.$$

This expression also does not depend on P; it is for given A, B, C a tensor of the second rank which is alternating or antisymmetric in the variables x, y, i. e.

$$\beta(x,y) = -\beta(y,x).$$

In the case of a plane, $\beta(x,y)$ is simply the area of the triangle ABC multiplied by the so-called alternating product of the vectors x and y; in the case when $N=3$, i. e. in ordinary space, $\beta(A,B,C;x,y)$ can be presented in the form of the scalar product of a certain vector v by the vector product of x and y; this vector v is perpendicular to the triangle ABC and its length is equal to the area of this triangle; it is called the vector area of the triangle or a bivector; the vector area of a triangle as well as its generalization for any contour have been considered by Peano, Laisant and Fréchet (see the latter's paper "Sur une généralisation des notions d'aire et de plan," *Nouv. An.* (4), 4 (1904), pp. 241-248); the present writer used it in connection with integration in space (see § 27 below).

When $N > 3$ it is impossible to get rid of the variable vectors x and y as in the cases $N = 2$ and $N = 3$ so that we have to consider the thing as a function of x and y. Schouten uses the term bivector also in this case. (Compare e. g. J. A. Schouten and D. J. Struik, "Einführung in die neueren Methoden der Differentialgeometrie," I, § 1.) It should be noticed that making $P = A$ we obtain a simpler but less symmetrical expression for β, viz.

$$24.11 \qquad \beta(A,B,C;x,y) = \tfrac{1}{2}\begin{vmatrix} (B-A).x & (B-A).y \\ (C-A).x & (C-A).y \end{vmatrix}.$$

If we are given a rectilinear contour $A, B, C, D, \cdots, K, L$ or a number of points in a given cyclic order, we form

$$24.2 \qquad \begin{aligned} \beta(A,B,C,\cdots,K,L,A;x,y) &= \beta(A,B,P;x,y) \\ &+ \beta(B,C,P;x,y) + \beta(C,D,P;\ x,y) + \cdots \\ &+ \beta(K,L,P;\ x,y) + \beta(L,A,P;\ x,y), \end{aligned}$$

an expression which, obviously, does not depend on P; we call this the tensor area of the contour $ABCD \cdots KL$. Given now a curvilinear contour Γ, we may approximate it by rectilinear contours and define the tensor area of Γ, which we shall denote by $\beta(\Gamma; x, y)$, as the limit of the tensor areas of these approximation contours when this limit exists, which happens in a very general case.

The notion may be extended by starting with four, or any number of points, instead of three; we have for the "tensor volume" of a tetrahedron $ABCD$ the expression

$$24.3 \qquad \gamma(A,B,C,D;x,y,z) = \tfrac{1}{6}\begin{vmatrix} 1 & (A-P).x & (A-P).y & (A-P).z \\ 1 & (B-P).x & (B-P).y & (B-P).z \\ 1 & (C-P).x & (C-P).y & (C-P).z \\ 1 & (D-P).x & (D-P).y & (D-P).z \end{vmatrix}.$$

If we have any number of points (> 4) in a given spherical order (as analogous to a cyclic order) we may form the tensor volume for such a polyhedron and by a limit process extend the notion to apply to a closed surface etc.

§ 25. *Contour variation of normals and a new introduction of the Riemann tensor.*

We introduced the indicator with the aid of the change $b - a$ of the

normal unit vector; instead of the vector $b - a$ we may here again write the tensor of the first rank $(b - a).x$ or

25.1 $$\begin{vmatrix} 1 & a.x \\ 1 & b.x \end{vmatrix} .$$

If $N - n = r > 1$ this cannot be generalized immediately. We denote as before the tangent bundle at a point A by the same capital letter A, and the normal bundle we denote by the corresponding small letter a. The projection of a vector x on the normal bundle we denote by affixing this letter a: thus x_a; in the case $r = 1$ this would be $a(a.x)$. The expression 25.1 suggests the formation of

25.2 $$\delta(A, B, C; x, y) = \begin{vmatrix} 1 & x_a & y_a \\ 1 & x_b & y_b \\ 1 & x_c & y_c \end{vmatrix}$$

when we are given three points A, B and C; this is a tensor of the N.-E.S. which characterizes the relative positions of the normal r-dimensional Euclidean spaces to one curved n-S. at the three points A, B, and C. It is, like the tensor area, an alternating tensor and, exactly as in the case of the tensor area, the definition of this tensor, which we call *the variation of normals* (normal Euclidean spaces or normal bundles) may be extended to apply to any rectilinear, and then to a curvilinear contour by first forming the expression

$$\delta(A, B, P; x, y) + \delta(B, C, P; x, y) + \cdots + \delta(L, A, P; x, y)$$

and then passing to the limit.

We thus obtain for a contour Γ a tensor $\delta(\Gamma; x, y)$ *the variation of normals* to the n-S. *for the contour* Γ. This corresponds to the finite change of the unit normal vector between two points.

Now, according to our program, we pass to the infinitesimal variation, which corresponds to an infinitesimal contour. We begin with a triangular contour A, B, C considering A as a fixed point and B and C as variable points approaching A in such a way that $\frac{B - A}{\lambda} \to h_A$, $\frac{C - A}{\mu} \to k_A$. We write 25.2 in the form

25.21 $$\delta(A, B, C; x, y) = \begin{vmatrix} x_b - x_a & y_b - y_a \\ x_c - x_a & y_c - y_a \end{vmatrix} .$$

If x and y are vectors of the tangent bundle at A we have $x_a = 0$, $y_a = 0$;

we may write in this case x_A for x and y_A for y; x_b will be $x_A - x_{A.B}$ etc., so that

$$\delta(A, B, C; x_A, y_A) = \begin{vmatrix} x_{A.B} - x_A & y_{A.B} - y_A \\ x_{A.C} - x_A & y_{A.C} - y_A \end{vmatrix} .$$

Dividing both parts by $\lambda\mu$ and passing to the limit we have (§ 17, Second paper) on the right hand side the Riemann tensor at the point A. A connection is thus established between the infinitesimal variation of normals and the Riemann tensor.

But this is not the whole story; the Riemann tensor gives us only *those* values of the tensor expressing the infinitesimal variation which correspond to the values of x and y *belonging to the curved space.* In order to know the infinitesimal variation of the normals completely we have to consider not only $\delta(x_A, y_A)$ but $\delta(x, y)$ for *all* values x, y. If we write in the general case $x = x_A + x_a$, $y = y_A + y_a$, we have

$$x_b - x_a = x_A - x_{A.B} + x_{ab} - x_a,$$

where x_{ab} denotes the projection of the normal component (with respect to the point A) x_a of x on the normal bundle at B; but then $x_{ab} - x_a = -(x_a)_B$ and we have (23.6)

$$\lim \frac{(x_a)_B}{\lambda} = -\mathfrak{F}(x_a, h)$$

The full expression for the infinitesimal variation is therefore

$$\begin{vmatrix} s(x_A, h_A) + \mathfrak{F}(x_a, h_A) & s(x_A, k_A) + \mathfrak{F}(x_a, k_A) \\ s(y_A, h_A) + \mathfrak{F}(y_a, h_A) & s(y_A, k_A) + \mathfrak{F}(y_a, k_A) \end{vmatrix} .$$

Since s and $\mathfrak{F}$ are perpendicular to each other, s being a normal vector and $\mathfrak{F}$ a tangential one (see 23.5), the products of the type $s.\mathfrak{F}$ vanish and we only have

$$25.4 \qquad \begin{vmatrix} s(x_A, h_A) & s(x_A, k_A) \\ s(y_A, h_A) & s(y_A, k_A) \end{vmatrix} + \begin{vmatrix} \mathfrak{F}(x_a, h_A) & \mathfrak{F}(x_a, k_A) \\ \mathfrak{F}(y_a, h_A) & \mathfrak{F}(y_a, k_A) \end{vmatrix} .$$

The first term is the Riemann tensor. We see thus that the Riemann tensor gives us only *a part* of the infinitesimal variation of normals; we may say that it gives all the information concerning the variation which can be expressed in terms of the curved space, of its *internal* properties, because the other part involves vectors x_a, y_a which require the consideration of the containing space, i. e. which refer to external properties (compare § 17, Second paper).

Here we have a radical difference between the Riemann tensor in general space and the indicator of a surface: the latter gives us full information concerning the change of the unit normal vector because this change can be controlled from the surface; the variation of normals cannot be controlled from the space (in general) and the Riemann tensor, which is an internal thing, only gives us partial information.

But there are cases where the *whole* variation can be given by the Riemann tensor; this happens obviously, when the second part of 25.4 vanishes, i. e. for spaces for which we have

$$25.5 \qquad \mathfrak{F}(x_a, h) \cdot \mathfrak{F}(y_a, k) = \mathfrak{F}(x_a, k) \cdot \mathfrak{F}(y_a, h);$$

in these cases, we may say, the whole infinitesimal variation lies in the curved space and is, therefore, given by the Riemann tensor. Only in these cases can we hope to obtain the finite variation of normals, which corresponds to a finite contour, by integrating the Riemann tensor in analogy to what we had in the case of a surface.

We shall not investigate here the peculiarities of spaces for which condition 25.5 is satisfied; this would lead us to considerations which lie outside the scope of the present paper; in dealing with surface integrals we will, however, again be led to relation 25.5.

We want now to get rid of the restriction which we imposed upon ourselves by considering only *triangular* infinitesimal contours. Let us consider a rectilinear infinitesimal contour, constituted by the points $A + h$, $A + k$, $A + l$, $A + m, \cdots, A + n$, $A + h$. It is clear from the preceding that the corresponding variation of normals is given by $\rho(x, y; h, k) + \rho(x, y; k, l) + \rho(x, y; l, m) + \cdots + \rho(x, y; n, h)$.

From the expression 17.2 for the Riemann tensor we conclude that, for fixed x and y, $\rho(x, y; h, k)$ is a sum of terms of the type

$$\begin{vmatrix} ph & pk \\ qh & qk \end{vmatrix}$$

where p and q are vectors belonging to the bundle A. The infinitesimal variation of normals which we are considering has, therefore, for its expression

$$25.6 \qquad \Sigma \left\{ \begin{vmatrix} ph & pk \\ qh & qk \end{vmatrix} + \begin{vmatrix} pk & pl \\ qk & ql \end{vmatrix} + \begin{vmatrix} pl & pm \\ ql & qm \end{vmatrix} + \cdots \begin{vmatrix} pn & ph \\ qn & qh \end{vmatrix} \right\}$$

where the summation is extended over a certain number of pairs of vector p_1, q_1, p_2, q_2 etc.

Now the tensor area of the contour $A+h$, $A+k$, $\cdots$, $A+n$ is

$$25.7 \qquad \begin{vmatrix} xh & xk \\ yh & yk \end{vmatrix} + \begin{vmatrix} xk & xl \\ yk & yl \end{vmatrix} + \begin{vmatrix} xl & xm \\ yl & ym \end{vmatrix} + \cdots \begin{vmatrix} xn & xh \\ yn & yh \end{vmatrix}.$$

Suppose that we take another contour *with the same tensor area;* this means that 25.7 has for the new values of $h, k, l, \cdots, m, n$ and for *arbitrary* values of x, y, the same value as for the original $h, k, l, \cdots, m, n$. Making x, y in turn equal to p_1, q_1; p_2, q_2 etc. we see that all the terms of 25.6 do not change when we substitute new values of $h, k, l, \cdots, m, n$ for the old ones, provided the new contour has the same tensor area. *The infinitesimal change of normals for an infinitesimal contour* Γ *in the neighborhood of a point A of the n-S. is thus seen to depend solely on the tensor area of this contour;* and of course it is a linear function of this tensor area. This completes the analogy with the indicator considered as a linear function of the displacement vector giving the infinitesimal change of the unit normal vector.

We may add that this gives a plausible interpretation of the Riemann tensor as a *bivector tensor* as Schouten puts it (loco citato, p. 34).

§ 26. *Contraction.*

In order to deal with integrals extended over surfaces, volumes etc. we must now introduce the operation of contraction, the use of which we have been able to avoid hitherto.

We start with the identity

$$26.1 \qquad x = \sum_{\nu=1}^{n} i_\nu (i_\nu x)$$

where $i_1, i_2, \cdots, i_n$ are n mutually perpendicular unit vectors of an n-dimensional bundle and which holds for every vector x of the bundle. We shall have occasion to use expressions of this type, i. e. sums of n terms each term of which contains twice one of the n mutually perpendicular unit vectors; if we have a tensor or a multilinear vector function $\phi(x, y, z, \cdots)$, the expression which we have in mind is

$$\sum_{\nu=1}^{n} \phi(i_\nu, i_\nu, z, \cdots).$$

It is easy to show that such an expression does not depend upon the particular set of n mutually perpendicular unit vectors which we use, i. e. that it does

not change if we substitute one such set for another. We therefore introduce for such a sum the notation $\phi(I, I, z \cdots)$ replacing each of the two variable vectors for which i_ν stand in the sum by a Roman numeral. If such an operation has to be performed more than once on the same tensor we use each time another numeral, so that $\phi(I, II, I, II)$ stands for

$$\sum_{\nu=1}^{n} \phi(I, i_\nu, I, i_\nu) .$$

This operation is essentially the same as the operation of contraction (Verjüngung) of tensors in the usual form of tensor calculus with the difference that we have here no distinction between lower and upper indices. The Roman numerals stand here for the dummy suffices.

With the notation introduced, identity 26.1 takes the form

26.11 $$x = I(Ix)$$

multiplying this by y we obtain

26.2 $$xy = (Ix)(Iy)$$

which means that the product of two vectors is the sum of the products of their components.

If we have a scalar linear function $\phi(x)$—a tensor of rank one—we may write

26.3 $$\phi(x) = \phi[I(Ix)] = \phi(I)(Ix) = [\phi(I).I].x .$$

The expression $\phi(I)I$ is a vector which written explicitly is

$$\phi(I)I = \sum_{\nu=1}^{n} \phi(i_\nu)\, i_\nu$$

if we denote this vector by g, we have $\phi(x) = g.x$. The operation of contraction permits us thus to write explicitly in terms of a scalar linear function the vector which is the constant factor of it (compare property 1 § 3, First paper).

It may be said that our definition of contraction contains traces of a coordinate system; this may be avoided to a certain extent by using the vector product of $n-1$ vectors but the expression, though more intrinsic, becomes more cumbersome and it was not thought worth while to introduce it here. We only shall mention that in the case of 3 dimensions we have

$$\phi(I, I).(\overline{xyz}) = \phi(x, y \hat{} z) + \phi(y, z \hat{} x) + \phi(z, x \hat{} y)$$

where $x \hat{} y$ denotes the vector product and $\overline{xyz}$ the triple scalar product, and

x, y, z are any three not co-planar vectors and in the case of four dimensions we have similarly,

$$\phi(I,I)\,.\,\overline{(xyzu)} = \phi(\overline{xyz},\ u) - \phi(\overline{yzu},\ x) + \phi(\overline{zux},\ y) - \phi(\overline{uxy},\ z)$$

where $\overline{xyz}$ is the vector product and $\overline{xyzu}$ the quadruple scalar product.

The operation of contraction may be used to pass from one form of the indicatrix to another; for instance (23.5)

$$\mathfrak{F}(a,x) = \psi(x,I,a)\,.\,I = [a\,.\,s(x,I)]\,.\,I.$$

We are not going to enter into the details concerning the products mentioned above; the vector algebra in 4 as well as in 3 dimensions can be developed intrinsically (see e. g. a paper by A. Mochoolsky in the Memoirs of the Research Institute, Odessa, February, 1924 (in Russian)); here for the sake of completeness we only shall state that, in three dimensions the vector product $x\hat{}y$ is the vector whose components are the determinants of the matrix

$$\left|\left|\begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix}\right|\right|$$

the triple scalar product $\overline{xyz}$ in three dimensions is the three row determinant formed with the components of x, y, z; the vector product $\overline{xyz}$ in four dimensions is the vector whose components are determinants of the matrix

$$\left|\left|\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{matrix}\right|\right|$$

and the scalar quadruple product is the four-row determinant of the components of four vectors.

§ 27. *Surface Integrals in Curved Space.*

If we have in a n-S. an alternating (antisymmetric) tensor field of the second rank $\phi(A, x_A, y_A)$ and a surface Σ, we may integrate this field over the surface. If we imagine the surface divided into "infinitesimal triangles" so that two of the sides of each triangle may be considered as displacements of one of its vertices A, we form the sum of expressions $\phi(A, dA, \delta A)$ and this sum might be considered as the surface integral. But it is more satisfactory to consider any finite division of the surface by means of a net of curves on it. We form the tensor area $\beta(\Gamma_\nu;\ x, y)$ for the contour which

constitutes the frontier of each part, we multiply this by the value $\phi(A_\nu, x, y)$ of the given tensor field at some point A_ν belonging to the same part of the surface and contract this product so as to obtain

$$\phi(A_\nu, I, II) \cdot \beta(\Gamma_\nu; I, II);$$

we form the sum of these expressions for all the parts into which the surface was divided and consider the integral as the limit (if any) to which these sums tend when the diameters of the parts (i. e. the upper limits of the distances between two points belonging to the same part) tend toward zero.

It is not the purpose of this paper to enter into details concerning the definition of integrals and the conditions under which they exist; we are interested here in questions of a less subtle nature; it might be interesting, however, to notice that the above definition of a surface integral *does not suppose the existence of tangent planes to the surface.* Integration in ordinary space was considered from a similar point of view in a paper by the present writer which appeared (in Russian) in the *Bulletin of the Physico-Mathematical Society of Kazan* in 1915.

If we consider the surface in a *Euclidean* space and have in this space a tensor field of a rank higher than two, which is alternating in two arguments, say $\phi(A, x, y, z, \cdots)$, with

$$\phi(A, x, y, z, \cdots) = -\phi(A, y, x, z, \cdots),$$

we may integrate it over the surface by giving to the "superfluous" arguments z etc. fixed values. But if we consider the surface in a *curved* n-dimensional space this is not possible and we have to resort to the device, used in § 21, of giving to z and other arguments values which are projections of fixed vectors in the containing N.-E.S. We obtain thus as the result of integration of a tensor field of rank k (which is alternating in two of its arguments) over a surface a tensor of the containing space of rank $k-2$.

Integration over a volume, i. e. a spread of 3 dimensions presents after this no difficulties: the integrand must be a tensor field alternating in three of its arguments; we divide the volume in parts; combine the tensor volume of the frontier of each part with the value of the integrand at some point of this part, by multiplying and then contracting three times; we give to superfluous arguments values which are projections of fixed vectors of the containing space; form the sum of the numbers so obtained; and pass to the limit making the diameters of the parts tend toward zero. The integral is a tensor of rank $k-3$ of the containing space.

§ 28. *Integrability Conditions for Surface Integrals.*

We shall take over from the usual coordinate exposition, similarly to what was done in § 22, the integrability condition for the surface integral of a tensor field of rank two. This condition, viz., the condition that an integral of $\phi(A, x_A, y_A)$ over a surface depends only on the contour of the surface, i. e. that it is the same for two surfaces having for their frontiers the same contour provided that one can be continuously transformed into the other without leaving the part of the n-S. in which $\phi(A, x_A, y_A)$ is considered, is that in this part ϕ satisfies the relation

28.1 $$\phi'(x, y, h) + \phi'(y, h, x) + \phi'(h, x, y) = 0,$$

where $\phi'(x, y, h)$ is the differential of $\phi(x, y)$.

If we are given a tensor field of higher rank than two, let us say, for example, a tensor field of rank four $\phi(x, y, z, u)$ which is alternating in the arguments x, y, and we integrate it with respect to these arguments using the method indicated in § 27 we have to write that the integrability conditions are satisfied for any tensor field $\omega(x, y)$ which is obtained from $\phi(x, y, z, u)$ with the aid of two vectors c, q of the N.-E.S. which contains the n-S. in which the tensor field ϕ and the surfaces over which we integrate are given, viz.,

$$\omega(A, x_A, y_A) = \phi(A, x_A, y_A, c_A, q_A).$$

Just as in § 23 we obtain the conditions

28.2 $$\phi[x, y, \mathfrak{F}(a, h), u] + \phi[y, h, \mathfrak{F}(a, x), u] + \phi[h, x, \mathfrak{F}(a, y), u] = 0$$

and

28.3 $$\phi[x, y, z, \mathfrak{F}(a, h)] + \phi[y, h, z, \mathfrak{F}(a, x)] + \phi[h, x, z, \mathfrak{F}(a, y)] = 0$$

which must be satisfied for all values of the normal vector a in addition to the differential condition

28.4 $$\phi'(x, y, z, u, h) + \phi'(y, h, z, u, x) + \phi'(h, x, z, u, y) = 0$$

where $\phi'(x, y, z, u, h)$ is the differential of $\phi(x, y, z, u)$.

Let us apply this to the Riemann tensor $\rho(x, y; z, u)$. The differential condition is always satisfied; in fact it is nothing but the Bianchi relation 12.5 (First paper) proved to hold in the general case toward the end of § 18 (Second paper). It is worth while to remember here that the differential condition for the indicator of a surface coincides with the Codazzi relation

which thus plays for the indicator the same rôle as the Bianchi relation plays for the Riemann tensor. We need only consider *one* of the algebraic conditions 28.2 and 28.3 since the variables z and u enter in ρ in the same way (only the sign is changed when they are interchanged so that 28.2 is equivalent to 28.3 in this case). If we use expression 17.3 (Second paper) for the Riemann tensor, 28.2 becomes

$$28.4 \quad \begin{vmatrix} s[x, \mathfrak{F}(a,h)] & s(x,u) \\ s[y, \mathfrak{F}(a,h)] & s(y,u) \end{vmatrix} + \begin{vmatrix} s[y, \mathfrak{F}(a,x)] & s(y,u) \\ s[h, \mathfrak{F}(a,x)] & s(h,u) \end{vmatrix} + \begin{vmatrix} s[h, \mathfrak{F}(a,y)] & s(h,u) \\ s[x, \mathfrak{F}(a,y)] & s(x,u) \end{vmatrix} = 0.$$

Expanding the determinants we obtain a sum of six products, the first of which is $s[x, \mathfrak{F}(a,h)] . s(y,u)$; we may identify this with the last term of 23.5 considering $s(y,u)$ as a, x as k_A, and $\mathfrak{F}(a,h)$ as h_A; using then the form of the middle term of identity 23.5 we may write our product as $\mathfrak{F}(a,h) . \mathfrak{F}[s(y,u), x]$; applying this to all the six products of the expansion of 28.4 and changing the order of terms we may write it as

$$\begin{aligned} \mathfrak{F}(a,h) . \mathfrak{F}[s(y,u),x] - \mathfrak{F}(a,x) . \mathfrak{F}[s(y,u),h] + \mathfrak{F}(a,x) . \mathfrak{F}[s(h,u),y] \\ - \mathfrak{F}(a,y) . \mathfrak{F}[s(h,u),x] + \mathfrak{F}(a,y) . \mathfrak{F}[s(x,u),h] \\ - \mathfrak{F}(a,h) . \mathfrak{F}[s(x,u),y] = 0. \end{aligned}$$

This vanishes when condition 25.5 *is satisfied;* this is very natural because in this case the Riemann tensor gives the full infinitesimal variation of normals and therefore its integral over a piece of surface gives the finite variation of normals for the frontier of this piece, i. e. for the contour which limits this piece and, of course, this does not depend upon the surface—just as the integral of the indicator along a curve depends only on its end points. In the general case the Riemann tensor does not give the full infinitesimal variation, its integral does not give therefore the finite variation and there is no reason why it should be determined by the contour alone.

As our next example we take the contracted Riemann tensor which we write, according to § 26, as $\rho(x, I; z, I)$. This is a symmetric tensor of rank two which we denote by $\epsilon(x,z)$; but instead of considering it as such, we make of it a tensor of rank n: we substitute for one of its arguments, say z, the vector product of $n-1$ vectors. If we take, for instance, $n=4$ we have to make $z = \overline{uvw}$, i. e. the vector product of the vectors u, v, w (see end of § 26).

Now we have in $\epsilon(x, \overline{uvw})$ a tensor which is alternating in the 3 arguments u, v, w. It may be therefore integrated over a volume (3-dimensional) and gives a vector of the containing space; we shall see what are the integra-

bility conditions, i. e. under what conditions the integral vanishes over every closed volume. If we write

$$\psi(u, v, w) = \epsilon(c_A, u, v, w)$$

the conditions in question are given by

$$\psi'(u, v, w, h) - \psi'(v, w, h, u) + \psi'(w, h, u, v) - \psi'(h, u, v, w) = 0.$$

This again gives a differential condition for ϵ viz.,

$$\epsilon'(x, \overline{uvw}, h) - \epsilon'(x, \overline{vwh}, u) + \epsilon'(x, \overline{whu}, v) - \epsilon'(x, \overline{huv}, w) = 0$$

which may be written as

$$\epsilon'(x, I, I) = 0$$

and an algebraic condition, viz.,

$$\epsilon[\mathfrak{F}(a, h), \overline{uvw}] - \epsilon[\mathfrak{F}(a, v), \overline{vwh}] + \epsilon[\mathfrak{F}(a, v), \overline{whu}] - \epsilon[\mathfrak{F}(a, w), \overline{huv}] = 0$$

or

$$\epsilon[\mathfrak{F}(a, I), I] = 0.$$

The discussion of spaces where these equations are satisfied will be given elsewhere.

§ 29. *Conclusion. Integral Expressions Which do Not Require the Consideration of the Containing Space.*

We had to go out of the space in order to consider integrals in the preceding sections as soon as the integrand had superfluous arguments, i. e. more than one in the case of a line integral, more than two (in which the tensor is antisymmetric) in the case of a surface integral, and so on. But it is desirable to be able to form in such cases also integral expressions not involving the conideration of the containing space. In some cases it is possible. Let us begin again with line integrals and let us take as the integrand a tensor field of the second rank which is symmetric in its arguments, so that we have

29.1 $$\phi(x, y) = \phi(y, x).$$

It can be proved that such a tensor can be presented in the form

29.2 $$\phi(x, y) = \pm (ax)(ay) \pm (bx)(by) \pm (cx)(cy) + \cdots$$

where a, b, c, $\cdots$ are mutually perpendicular vectors, the number of terms

being equal to the number of dimensions of the space and the $\pm$ signs being used in order to avoid the introduction of imaginary vectors. Now the products ax, bx, cx, etc. are tensors of the first rank; they can be integrated, therefore, without leaving the curved space; we introduce the notations

$$29.3 \qquad P = \int a.dA, \qquad Q = \int b.dA, \qquad R = \int c.dA, \quad \cdots$$

and form the expression

$$29.4 \qquad \pm P^2 \pm Q^2 \pm R^2 \pm \cdots$$

using the same signs as in the corresponding places in 29.2. The expression 29.4 is the sought integral expression. Let us consider as an example the case of an ordinary (two-dimensional) surface and as the integrand let us take the indicator; if i, j denote the two unit vectors which belong to the principal directions and λ, μ are the curvatures we have $a = i\sqrt{\lambda}$, $b = j\sqrt{\mu}$ so that it will be convenient to introduce the notations $\sqrt{\lambda} = \alpha$, $\sqrt{\mu} = \beta$. Let us ask ourselves when the integrals P and Q are independent of the path. Since the integrand of P is $\omega(x) = a.(ix)$ the condition 22.1 takes the form

$$29.51 \qquad a'(h).(ix) + a[i'(h).x] = a'(x).(ih) + a.[i'(x).h];$$

for Q we have

$$29.52 \qquad \beta'(h).(jx) + \beta[j'(h).x] = \beta'(x).(jh) + \beta.[j'(x).h].$$

On the other hand the Codazzi equation for the indicator, which now has the form

$$\alpha^2(ix)(iy) + \beta^2(jx)(jy),$$

gives

$$\begin{aligned} &2a.a'(h)(ix)(iy) + a^2[i'(h).x](iy) + a^2(ix)[i'(h).y] \\ &\quad + 2\beta.\beta'(h)(jx)(jy) + \beta^2[j'(h).x].(jy) + \beta^2(jx)[j'(h).y] \\ &= 2a.a'(x).(ih)(iy) + a^2[i'(x).h](iy) + a^2(ih)[i'(x).y] \\ &\quad + 2\beta.\beta'(x).(jh)(jy) + \beta^2[j'(x).h](jy) + \beta^2(jh)[j'(x).y]. \end{aligned} \qquad 29.6$$

The simultaneous existence of equations 29.51, 29.52 and 29.6 leads to the relation (see for the method used the author's paper in the January number of the *Transactions of the American Math. Society* for 1925, p. 123),

$$\alpha^2 + \beta^2 = 0 \quad \text{or} \quad \lambda + \mu = 0$$

so that only for a minimal surface are the integrals P and Q, and therefore the expression 29.4, independent of the path. These questions connected with minimal surfaces will be treated in detail elsewhere.

Let us consider now the case of a surface integral and a tensor field of rank four $\phi(x, y; u, v)$ with the property

$$\phi(x, y; u, v) = \phi(u, v; x, y)$$

as the integrand in a 4-S. Suppose that ϕ may be presented in the form

$$\lambda \begin{vmatrix} ix & iy \\ jx & jy \end{vmatrix} \cdot \begin{vmatrix} iu & iv \\ ju & jv \end{vmatrix} + \mu \begin{vmatrix} kx & ky \\ lx & ly \end{vmatrix} \cdot \begin{vmatrix} ku & kv \\ lu & lv \end{vmatrix}$$

Precisely such a tensor occurs in physics, viz. the *second part* of the Riemann tensor of space-time. It is natural, in order to find an integral expression to put again $\sqrt{\lambda} = \alpha$, $\sqrt{\mu} = \beta$ and to consider the expression

$$P^2 + Q^2$$

where

$$P = \int \alpha \begin{vmatrix} i.dA & i.\delta A \\ j.dA & j.\delta A \end{vmatrix}, \qquad Q = \int \beta \begin{vmatrix} k.dA & k.\delta A \\ l.dA & l.\delta A \end{vmatrix}.$$

This expression has in certain cases a physical meaning, being connected with the square of the electric charge; in other cases it is possible to obtain by a slight modification another more general integral expression of the same type. We shall not go into details concerning this type of integral expressions. The purpose of this section was only to indicate by way of conclusion, that the integrals involving the consideration of vectors of the containing space are not the only integral expressions which can be formed with tensor fields having superfluous arguments.

BALTIMORE, JUNE 14, 1925.

On Certain Symmetric Sums of Determinants.

By Lloyd L. Dines.

The first theorem of this paper has to do with a certain transformation of determinants which may be defined as follows. If D is a determinant of order n, and if $a_1, a_2, \cdots, a_n$ is any sequence of numbers (not necessarily distinct) each chosen from the set $0, 1, 2, \cdots, n$, then by the notation

$$T_{a_1 a_2 \cdots a_n} D$$

we shall mean the determinant of nth order obtained from D by shifting each element of the first column up a_1 places (thus eliminating the upper a_1 of them from the determinant), shifting each element of the second column up a_2 places, and so on to the nth column, and then filling in the vacant places at the bottom by zeros. From the definition it is obvious that $T_{00 \cdots 0} D \equiv D$, while if every a_i is positive or if any a_i is equal to n, then $D_{a_1 a_2 \cdots a_n} D = 0$. Between these extremes are the cases of interest.

For a given sequence $a_1, a_2, \cdots, a_n$, we shall consider all the distinguishable * permutations of the sequence.

By the notation

$$ST_{a_1 a_2 \cdots a_n} D$$

we shall mean the sum of all the determinants $T_{a_{i_1} a_{i_2} \cdots a_{i_n}} D$ where $a_{i_1}, a_{i_2}, \cdots, a_{i_n}$ is a permutation of $a_1, a_1, \cdots, a_n$. The theorem † to be proved in § 1 is then to the effect that *if at least one element of the sequence* $a_1, a_2, \cdots, a_n$ *is positive*

$$ST_{a_1 a_2 \cdots a_n} D \equiv 0.$$

For example for $n = 3$, we have

$$ST_{001} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & b_3 \\ b_1 & b_2 & c_3 \\ c_1 & c_2 & 0 \end{vmatrix} + \begin{vmatrix} a_1 & b_2 & a_3 \\ b_1 & c_2 & b_3 \\ c_1 & 0 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & a_2 & a_3 \\ c_1 & b_2 & b_3 \\ 0 & c_2 & c_3 \end{vmatrix} \equiv 0,$$

* If no two elements of the sequence are equal then all its permutations are distinguishable. If certain elements of the sequence are equal, permutations which differ only by the interchange of equal elements are not distinguishable. In this case only one representative of the set of non-distinguishable permutations will be considered.

† Theorems somewhat similar to this one and to its generalization stated in § 1 have been given by Sylvester, Hammond, and Deruyts. See Muir's *History of Determinants*, Vol. IV, pages 2 and 25.

and other vanishing sums corresponding to the sequences (0, 0, 2); (0, 1, 1); and (0, 1, 2).

The theorem of § 2 has to do with sums of determinants of nth order from a persymmetric * array of n rows, the elements however being affected by binomial coefficients * according to a definite law.

For $n = 3$ the theorem would have reference to a persymmetric array

$$\begin{matrix} c_1 & c_2 & c_3 & \cdots \\ c_2 & c_3 & c_4 & \cdots \\ c_3 & c_4 & c_5 & \cdots \end{matrix}.$$

Denoting the binomial coefficients by the notation

$$\frac{m(m-1)\cdots(m-k+1)}{1\cdot 2\cdots k} = (m)_k,$$

we may state a simple case of the theorem as follows:

$$\sum_{j+k+l=p} \begin{vmatrix} (j+1)_1 c_{j+1} & (k+2)_2 c_{k+2} & (l+3)_3 c_{l+3} \\ (j+2)_2 c_{j+2} & (k+3)_3 c_{k+3} & (l+4)_4 c_{l+4} \\ (j+3)_3 c_{j+3} & (k+4)_4 c_{k+4} & (l+5)_5 c_{l+5} \end{vmatrix}$$

$$= \sum_{j+k+l=p} (j+3)_3 (k+4)_4 (l+5)_5 \begin{vmatrix} c_{j+1} & c_{k+2} & c_{l+3} \\ c_{j+2} & c_{k+3} & c_{l+4} \\ c_{j+3} & c_{k+4} & c_{l+5} \end{vmatrix},$$

the identity holding for every non-negative integer p.

§ 1.

Theorem: If D is a determinant of order n, and $a_1, a_2, \cdots, a_n$ is a sequence of numbers (not necessarily all distinct) each chosen from the set $0, 1, \cdots, n$, and at least one of them positive, then

$$ST_{a_1 a_2 \cdots a_n} D \equiv 0.$$

For $n = 2$ the theorem is obviously true since

$$ST_{01} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \equiv \begin{vmatrix} a_{11} & a_{22} \\ a_{21} & 0 \end{vmatrix} + \begin{vmatrix} a_{21} & a_{12} \\ 0 & a_{22} \end{vmatrix} \equiv 0.$$

Assuming the theorem to be true for $n = r - 1$, we will prove it true

* For literature on persymmetric determinants and on determinants whose elements are binomial coefficients, see Muir's *History of Determinants*, Vol. IV, Chapter XIII, and Vol. III, Chapter XX.

for $n = r$, thus establishing its general validity by mathematical induction.

If we denote by $D^{(ij)}$ the algebraic complement of the element a_{ij} in the determinant

$$D \equiv |\, a_{ij} \,| \qquad \begin{matrix} i = 1, 2, \cdots, r, \\ j = 1, 2, \cdots, r \end{matrix}$$

we find upon consideration of the expansion of D according to elements of its first row, that

$$(1) \qquad ST_{a_1 a_2 \cdots a_n} D \equiv \sum_{k=1}^{r}{}' \sum_{j=1}^{r} a_{1+a_k, j}\, ST_{a_1 a_2 \cdots a_{k-1} a_{k+1} \cdots a_r} D^{(1j)},$$

the notation $\sum_{k=1}^{r}{}'$ indicating that in case a_k has the same value for two or more values of k, the summation as to k shall include only one term representing such values for each value of j.

Now with respect to the symmetric sum S on the right side of (1), there are two possibilities. First, if any one of the numbers $a_1, a_2, \cdots, a_{k-1}, a_{k+1}, \cdots, a_r$ is different from zero, then the symmetric sum vanishes on account of our assumption that our theorem is true for determinants of order $r - 1$. On the other hand, if $a_1 = a_2 = \cdots = a_{k-1} = a_{k+1} = \cdots = a_r = 0$ then by hypothesis $a_k \neq 0$. And for the value of k which gives rise to this situation, the right side of (1) reduces to

$$\sum_{j=1}^{r} a_{1+a_k, j}\, D^{(1j)},$$

which vanishes by an elementary theorem on determinants. Hence the entire right side of (1) vanishes, and our theorem is established.

A number of variations of the theorem immediately suggest themselves. For example, the "shifting" involved in the transformation T might obviously be made downward, to the right, or to the left, as well as upward. We will state one extension which is not quite so obvious, and which will be used in § 2.

With each element a_k of the sequence $a_1, a_2, \cdots, a_n$ we will associate a sequence of $n - a_k$ elements

$$(2) \qquad s_k = (s_{1k}, s_{2k}, \cdots, s_{n-a_k, k}) \quad (k = 1, 2, \cdots, n),$$

subject only to the restriction that if $a_i = a_j$, then $s_i \equiv s_j$. And we will denote by

$$T_{\substack{s_1 s_2 \cdots s_n \\ a_1 a_2 \cdots a_n}} D$$

the determinant obtained from D by shifting the elements of the kth column up a_k places, filling in the vacant places by zeros, and then multiplying the upper $n - a_k$ elements of the resulting column by the respective elements of the sequence s_k, this procedure to be followed for each column of D.

By the notation

$$ST_{\substack{s_1 s_2 \cdots s_n \\ a_1 a_2 \cdots a_n}} D$$

we shall mean the sum of all determinants

$$T_{\substack{s_{i_1} s_{i_2} \cdots s_{i_n} \\ a_{i_1} a_{i_2} \cdots a_{i_n}}}$$

where $a_{i_1}, a_{i_2}, \cdots, a_{i_n}$ is a permutation of $a_1, a_2, \cdots, a_n$. We may then state the generalization in the form:

If D is a determinant of order n, and $a_1, a_2, \cdots, a_n$ is a sequence of numbers chosen from the set 0, 1, 2, $\cdots$, n, of which not all elements are zero, and $s_1, s_2, \cdots, s_n$ are sequences of form (2); then

$$ST_{\substack{s_1 s_2 \cdots s_n \\ a_1 a_2 \cdots a_n}} D \equiv 0.$$

The proof may be made by an induction similar to that used in the simpler theorem.

§ 2.

In this section, we suppose the elements of the persymmetric array

$$\begin{array}{ccccc} c_1 & c_2 & c_3 & \cdot & \cdot \\ c_2 & c_3 & c_4 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_n & c_{n+1} & c_{n+2} & \cdot & \cdot \end{array}$$

to be well defined for as many columns as the statement of the theorem requires.

Denoting the binomial coefficients by the notation

$$(m)_k = \frac{m(m-1)\cdots(m-k+1)}{1 \cdot 2 \cdots k},$$

we will prove the following

Theorem: For any non-negative integer p, and any sequence of non-negative integers $r_1, r_2, \cdots, r_n$,

$$\sum_{j_1+j_2+\cdots+j_n=p} \begin{vmatrix} (j_1+r_1+1)_{r_1+1}c_{j_1+r_1+1} & (j_2+r_2+1)_{r_2+1}c_{j_2+r_2+1} & \cdots & (j_n+r_n+1)_{r_n+1}c_{j_n+r_n+1} \\ (j_1+r_1+2)_{r_1+2}c_{j_1+r_1+2} & (j_2+r_2+2)_{r_2+2}c_{j_2+r_2+2} & \cdots & (j_n+r_n+2)_{r_n+2}c_{j_n+r_n+2} \\ \cdot & \cdot & \cdot\cdot & \\ \cdot & \cdot & \cdot\cdot & \\ \cdot & \cdot & \cdot\cdot & \\ (j_1+r_1+n)_{r_1+n}c_{j_1+r_1+n} & (j_2+r_2+n)_{r_2+n}c_{j_2+r_2+n} & \cdots & (j_n+r_n+n)_{r_n+n}c_{j_n+r_n+n} \end{vmatrix}$$

$$= \sum_{j_1+j_2+\cdots+j_n=p} (j_1+r_1+n)_{r_1+n}(j_2+r_2+n)_{r_2+n}\cdots(j_n+r_n+n)_{r_n+n}$$

$$\times \begin{vmatrix} c_{j_1+r_1+1} & c_{j_2+r_2+1} & \cdots & c_{j_n+r_n+1} \\ c_{j_1+r_1+2} & c_{j_2+r_2+2} & \cdots & c_{j_n+r_n+2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ c_{j_1+r_1+n} & c_{j_2+r_3+n} & \cdots & c_{j_n+r_n+n} \end{vmatrix} .$$

For the proof we shall make use of the following easily deducible relation between binomial coefficients:

$$(j+r+h)_{r+h}$$
$$\equiv (j+r+n)_{r+n} - (n-h)_1(j+r+n-1)_{r+n} + (n-h)_2(j+r+n-2)_{r+n} - \cdots \pm (j+r+h)_{r+n},$$

which holds for $h = 1, 2, \cdots, n$.

By means of this relation, the elements of any column in (3) may be replaced respectively by polynomials

$$(j+r+n)_{r+n}c_{j+r+1} - (n-1)_1(j+r+n-1)_{r+n}c_{j+r+1} + \cdots \pm (j+r+1)_{r+n}c_{j+r+1}$$
$$(j+r+n)_{r+n}c_{j+r+2} - (n-2)_1(j+r+n-1)_{r+n}c_{j+r+2} + \cdots \mp (j+r+2)_{r+n}c_{j+r+2}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$(j+r+n)_{r+n}c_{j+r+n-1} - (1)_1(j+r+n-1)_{r+n}c_{j+r+n-1}$$
$$(j+r+n)_{r+n}c_{j+r+n}$$

the symbols j and r carrying subscripts indicative of the particular column considered.

Each determinant of the sum (3) can therefore be replaced by the sum

of n^n constituent determinants (many of which obviously vanish) with the property that in each column all the elements contain a common binomial coefficient factor. The typical form of these columns will be

$$(4)\qquad \begin{matrix} (-1)^a(n-1)_a(j+r+n-a)_{r+n}c_{j+r+1} \\ (-1)^a(n-2)_a(j+r+n-a)_{r+n}c_{j+r+2} \\ \cdot \\ \cdot \\ \cdot \\ (-1)^a(a)_a\,(j+r+n-a)_{r+n}c_{j+r+n-a} \\ 0 \\ \cdot \\ \cdot \\ 0 \end{matrix}$$

the last a elements being zero. There will be such columns corresponding to $a = 0, 1, \cdots, n-1$, and we will say that the column (4) has the *character* a.

Consider now any set of numbers $a_1, a_2, \cdots, a_n$, each chosen from the sequence $0, 1, \cdots, n-1$, not necessarily distinct, but at least one of them being zero. Consider the constituent determinant in which the first column (involving the index j_1) has the character a_1, the second column (involving the index j_2) has the character a_2, and so on. We will say that this constituent determinant has the character $(a_1, a_2, \cdots, a_n)$. Out of this determinant we can take the factor

$$(5)\qquad (-1)^{a_1+a_2+\cdots+a_n}(j_1+r_1+n-a_1)_{r_1+n}(j_2+r_2+n-a_2)_{r_2+n}\cdots (j_n+r_n+n-a_n)_{r_n+n},$$

leaving a residual determinant of which a typical column is

$$\begin{matrix} (n-1)_a c_{j+r+1} \\ (n-2)_a c_{j+r+2} \\ \cdot \\ \cdot \\ \cdot \\ (a)_a\, c_{j+r+n-a} \\ 0 \\ \cdot \\ \cdot \\ 0 \end{matrix}$$

If $(a_{i_1}, a_{i_2}, \cdots, a_{i_n})$ is any permutation of $(a_1, a_2, \cdots, a_n)$ there will

be a constituent determinant having the character $(a_{i_1}, a_{i_2}, \cdots, a_{i_n})$, and out of it we can take the factor

$$(6)\quad (-1)^{a_1+a_2+\cdots+a_n}(j_1+r_1+n-a_{i_1})_{r_1+n}(j_2+r_2+n-a_{i_2})_{r_2+n}\cdots (j_n+r_n+n-a_{i_n})_{r_n+n},$$

which in general will be different from (5).

It now becomes necessary for us to note a significant property of the sum (3) which we have not previously stressed, and which will enable us to correlate the products (5) and (6). The notation

$$\sum_{j_1+j_2+\cdots+j_n=p}$$

indicates that the summation is to be made for every combination of indices $j_1, j_2, \cdots, j_n$ of which the sum is p. Hence if in a determinant affected by this summation sign we make a transformation of indices

$$j_1 = j'_1 + a_1, \quad j_2 = j'_2 + a_2, \cdots, \quad j_n = j'_n + a_n,$$

there will be no change in the content of the summation, *provided* $a_1 + a_2 + \cdots + a_n = 0$.

If therefore in the determinant with character $(a_{i_1}, a_{i_2}, \cdots, a_{i_n})$ we make the substitution

$$j_1 = j'_1 + a_{i_1} - a_1, \quad j_2 = j'_2 + a_{i_2} - a_2, \cdots, \quad j_n = j'_n + a_{i_n} - a_n,$$

the content of the summation will not be affected. And upon dropping the primes from the indices $j'_1, j'_2, \cdots, j'_n$ in the product (6) after transformation it is seen to be identical with the product (5).

By means of similar transformations we are able to reduce each determinant whose character is a permutation of $(a_1, a_2, \cdots, a_n)$ to a form such that the identical factor (5) can be taken from it.

It is not very difficult to see further that the residual determinants are obtainable by a transformation $T^{s_1 s_2 \cdots s_n}_{a_1 a_2 \cdots a_n}$ from one and the same determinant. We may in fact write the sum (as indicated by $\sum_{j_1+j_2+\cdots+j_n=p}$) of all the constituent determinants whose characters are permutations of $(a_1, a_2, \cdots, a_n)$ in the form

$$(7)\quad \sum_{j_1+j_2+\cdots+j_n=p}(-1)^{a_1+a_2+\cdots+a_n}(j_1+r_1+n-a_1)_{r_1+n}(j_2+r_2+n-a_2)_{r_2+n}\cdots (j_n+r_n+n-a_n)\, ST^{s_1 s_2 \cdots s_n}_{a_1 a_2 \cdots a_n} D,$$

where

$$D = \begin{vmatrix} c_{j_1+r_1+1-a_1} & c_{j_2+r_2+1-a_2} & \cdot & \cdot & c_{j_n+r_n+1-a_n} \\ c_{j_1+r_1+2-a_1} & c_{j_2+r_2+2-a_2} & \cdot & \cdot & c_{j_n+r_n+2-a_n} \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ c_{j_1+r_1+n-a_1} & c_{j_2+r_2+n-a_2} & \cdot & \cdot & c_{j_n+r_n+n-a_n} \end{vmatrix}$$

and

$$s_k = ((n-1)_{a_k}, (n-2)_k, \cdots, (a_k)_{a_k}).$$

We are now ready to draw our conclusion. If the set $(a_1, a_2, \cdots, a_n)$ contains an element different from zero, the sum (7) vanishes by the generalized theorem of § 1. Therefore the only non-vanishing part of the sum (3) is given by (7) for $a_1 = a_2 = \cdots = a_n = 0$. This value is evidently

$$\sum_{j_1+j_2+\cdots+j_n=p} (j_1+r_1+n)_{r_1+n}(j_2+r_2+n)_{r_2+n}\cdots(j_n+r_n+n)_{r_n+n} \times \begin{vmatrix} c_{j_1+r_1+1} & c_{j_2+r_2+1} & \cdots & c_{j_n+r_n+1} \\ c_{j_1+r_1+2} & c_{j_2+r_2+2} & \cdots & c_{j_n+r_n+2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ c_{j_1+r_1+n} & c_{j_2+r_2+n} & \cdots & c_{j_n+r_n+n} \end{vmatrix}$$

and this is the value as stated in the theorem.

UNIVERSITY OF SASKATCHEWAN.

A Direct Solution of Systems of Linear Differential Equations having Constant Coefficients.*

By J. A. Nyswander.

1. *Introduction.* The theory of linear differential equations having constant coefficients began with Euler, about the middle of the eighteenth century, and was continued during the latter part of the century by other mathematicians, the most noteworthy of whom were Gauss, Laplace, and Lagrange. All of these writers except Lagrange treated a single equation of low order; Lagrange, however, was able to solve those special systems of linear differential equations in a few dependent variables for which the roots of the characteristic equation $D(\lambda) = 0$ were distinct. During the interval of about seventy years between Lagrange and Weierstrass, various symbolic methods were devised for solving a single equation of the nth order, while for the general simultaneous system of equations a treatment was given only for the simple case in which $D(\lambda) = 0$ has distinct roots. In this period special systems of equations for which $D(\lambda) = 0$ has multiple roots were solved by regarding them as the limiting cases of corresponding systems of equations for which $D(\lambda) = 0$ has distinct roots. This method is unsatisfactory logically and practically impossible of application.

The first complete treatment of a system of differential equations in n dependent variables was given by Weierstrass.† He started with the system

$$x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad (i = 1, \cdots, n), \tag{1}$$

where $x'_i \equiv dx_i/dt$, and made linear transformations on certain bilinear forms obtained from (1), thus securing a simplified equivalent system of equations of the form

$$\begin{cases} \xi'_1 = \lambda_1\xi_1, \\ \xi'_2 = \lambda_1\xi_2 + \delta_2\xi_1, \\ \cdots\cdots\cdots\cdots \\ \xi'_j = \lambda_1\xi_j + \delta_j\xi_{j-1}, \end{cases} \tag{2}$$

* Read before the American Mathematical Society at Chicago, April 18, 1924.

† Weierstrass, *Collected Works*, Vol. II, pp. 74-75.

where λ_1 is a root of order j of the equation $D(\lambda) = 0$. The properties of $D(\lambda)$ pertinent to the problem were expressed in terms of the elementary divisors associated with $D(\lambda)$ and its minors. Although Weierstrass' process is logically sound it is indirect and tedious in application. Solutions of the problem were also given by Weber * and Frobenius.† Their methods are essentially the same in principle as that of Weierstrass and differ from it only in certain minor details.

A method of treatment ‡ which is entirely different in character from that of Weierstrass was given by Thomé. By differentiating a single one of the differential equations $n-1$ times and replacing the x'_i at each step by $\sum_{j=1}^{n} a_{ij}x_j$, Thomé arrived at a single differential equation of the nth order in one of the x_i, say x_1. This procedure implies that a certain determinant of the coefficients of $x_2, \cdots, x_n$ in the n differentiated equations is different from zero. Each of the n solutions of this differential equation of the nth order gives rise to a set of simultaneous linear equations in $x_2, \cdots, x_n$ which can be solved. If, however, this determinant is zero the procedure just indicated leads to a system of non-homogeneous differential equations which must be solved in the process of arriving at the solutions of system (1). Thomé's treatment, although simpler in theory and application than any of the earlier methods, is likewise indirect and laborious to apply.

The present paper treats the general system of differential equations by a method which is both elementary and direct, and which at the same time is much simpler of application than any of the previous methods. A general form for the solution of system (1) is obtained, from which, in any particular case, the desired solutions can be written at once in terms of the minors of the determinant $D(\lambda_i)$ and of their λ-derivatives.

2. *The Solutions when $D(\lambda) = 0$ has Distinct Roots.* Before solving the problem in its greatest degree of generality the two cases for which the characteristic equation $D(\lambda) = 0$ has distinct roots, and a triple root, respectively, will be treated.

Consider the simultaneous system of equations

$$x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad (i = 1, \cdots, n), \tag{1}$$

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* H. Weber, *Lehrbuch der Algebra*, Vol. II, pp. 41-42.

† Crelle's *Journal*, Vol. 86, pp. 42-72; 146-209.

‡ Crelle's *Journal*, Vol. 131, pp. 8-24; Vol. 133, pp. 1-17.

where the a_{ij} are given constants. Let the question be raised whether or not they have a solution of the form

$$x_i = A_i e^{\lambda t} \qquad (i = 1, \cdots, n), \tag{2}$$

in which the A_i and λ are constants. On substituting (2) in (1), it is found that if (2) is a solution, then the A_i and λ must satisfy the conditions

$$\begin{cases} (a_{11} - \lambda)A_1 + a_{12}A_2 + \cdots + a_{1n}A_n = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}A_1 + a_{n2}A_2 + \cdots + (a_{nn} - \lambda)A_n = 0. \end{cases} \tag{3}$$

Equations (3) are linear in the A_i and will have a common solution in which not all A_i are zero if, and only if, λ is a root of the characteristic equation

$$D(\lambda) = \begin{vmatrix} a_{11} - \lambda, & a_{12}, & \cdots, & a_{1n} \\ a_{21}, & a_{22} - \lambda, & \cdots, & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{n1}, & a_{n2}, & \cdots, & a_{nn} - \lambda \end{vmatrix} = 0. \tag{4}$$

Suppose that $D(\lambda) = 0$ has n distinct roots $\lambda_1, \cdots, \lambda_n$. For each of these n roots equations (3) have a solution for which not all the A_i are zero. Since for each $\lambda = \lambda_j$ at least one first minor of $D(\lambda_j)$ does not vanish, one can express the n solutions of (3) in the form

$$A_j = \delta_{j1i} c_i \qquad (j, i = 1, \cdots, n), \tag{5}$$

where the c_i are arbitrary constants and the δ_{j1i} are the first-minor cofactors in $D(\lambda_i)$ resulting from the solution of (3) for the A_j by determinants. In general, the second subscript k of δ_{jki} is used to indicate that δ_{jki} represents a kth minor cofactor (of order $n - k$) of the determinant $D(\lambda_i)$.

On substituting (5) in (2), one obtains the n solutions of system (1)

$$\begin{cases} x_1 = c_i \delta_{11i} e^{\lambda_i t} \qquad (i = 1, \cdots, n), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = c_i \delta_{n1i} e^{\lambda_i t}. \end{cases} \tag{6}$$

A proof of the fact that (6) form a fundamental set of solutions of (1) will be omitted here since a demonstration of this property for the general case will be given in Section 7.

3. *Solutions of System* (1) *when* $D(\lambda) = 0$ *has a Triple Root.* Let $\lambda_1, \lambda_4, \cdots, \lambda_n$ be the n roots of $D(\lambda) = 0$, $\lambda = \lambda_1$ being a triple root. The following cases, classified with respect to the properties of $D(\lambda_1)$, can arise:

I. All first and second-minor cofactors (δ_{j1k} and δ_{j2k}) of $D(\lambda_1)$ are zero, while at least one third-minor cofactor (δ_{j3k}) is not zero.

II. At least one δ_{i1k} is not zero.

III. All δ_{i1k} are zero, at least one δ_{i2k} is not zero, and at least one $\delta_{i1k}(\lambda)$ has a λ-derivative that does not vanish for $\lambda = \lambda_1$.

Since the λ-derivative of every $(j-1)$st minor of $D(\lambda)$ is a sum of jth minors of $D(\lambda)$, it is evident that

$$\left[\frac{d^3D(\lambda)}{d\lambda^3}\right]_{\lambda=\lambda_1} = \sum_{i,\,k} \delta_{i3k}.$$

Thus if $\lambda = \lambda_1$ is a triple root of $D(\lambda) = 0$, the case for which all $\delta_{i1k} =$ all $\delta_{i2k} =$ all $\delta_{i3k} = 0$ cannot exist. It would appear that under III there would arise the sub-case obtained by replacing the last hypothesis of III by the one that every

$$\left[\frac{d}{d\lambda}\,\delta_{i1k}(\lambda)\right]_{\lambda=\lambda_1} = 0\ ,$$

at least one

$$\left[\frac{d}{d\lambda}\,\delta_{i2k}(\lambda)\right]_{\lambda=\lambda_1} \neq 0.$$

That this cannot occur when $\lambda = \lambda_1$ is only a triple root of $D(\lambda) = 0$ is readily seen by considering $D(\lambda)$ when reduced to its normal form.*

CASE I. In Section 2 it was seen that if the system of differential equations

$$x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad (i = 1, \cdots, n), \tag{1}$$

has a solution of the form

$$x_i = A_i e^{\lambda_1 t} \qquad (i = 1, \cdots, n), \tag{2}$$

then the A_i must satisfy the equations

$$(a_{ii} - \lambda_1)A_i + \sum_{k=1}^{n} a_{ik}A_k = 0 \qquad (i = 1, \cdots, n;\ k \neq i). \tag{3}$$

Since by the hypothesis of this case $D(\lambda_1)$ is of rank $n-3$, the solution of (3) may be written in the form

$$A_i = \delta_{i31}c_1 + \delta_{i32}c_2 + \delta_{i33}c_3 \qquad (i = 1, \cdots, n), \tag{4}$$

where c_1, c_2 and c_3 are arbitrary constants, and the δ_{i3k} are the third-minor cofactors in $D(\lambda_1)$ obtained in the solution of system (3) by determinants.

* Bôcher, *Introduction to Higher Algebra*, p. 267.

One observes that in consequence of the nature of solution (4), at least two of the δ_{i3k} $(i = 1, \cdots, n)$ are identically zero for $k = 1, 2, 3$ respectively, while at least one of the δ_{i3k}, in each of these three sets, is different from zero. On substituting (4) in (2), one obtains as the three solutions of equations (1)

$$x_j = c_i \delta_{j3i} e^{\lambda_1 t} \qquad (j = 1, \cdots, n;\ i = 1, 2, 3). \tag{5}$$

The remaining $n - 3$ solutions of system (1) corresponding to $\lambda = \lambda_4, \cdots, \lambda_n$ are given by (6), of Section 2, for $i = 4, \cdots, n$ respectively.

CASE II. At least one δ_{i1k} is not zero. It will be shown that in this case the solutions of equations (1), associated with $\lambda = \lambda_1$, have the form

$$x_i = (A_i^{(1)} t^2 + A_i^{(2)} t + A_i^{(3)}) e^{\lambda_1 t} \qquad (i = 1, \cdots, n), \tag{6}$$

where the $A_i^{(j)}$ are constants. On substituting (6) in (1) and equating coefficients of t^2, t and t^0, one obtains as sufficient conditions that (6) shall be a solution of (1)

$$(a_{ii} - \lambda_1) A_i^{(1)} + \sum_{k=1}^{n} a_{ik} A_k^{(1)} = 0 \qquad (i = 1, \cdots, n;\ k \neq i), \tag{7}$$

$$(a_{ii} - \lambda_1) A_i^{(2)} + \sum_{k=1}^{n} a_{ik} A_k^{(2)} = 2A_i^{(1)} \qquad (i = 1, \cdots, n;\ k \neq i), \tag{8}$$

$$(a_{ii} - \lambda_1) A_i^{(3)} + \sum_{k=1}^{n} a_{ik} A_k^{(3)} = A_i^{(2)} \qquad (i = 1, \cdots, n;\ k \neq i), \tag{9}$$

It will now be shown that these conditions can be satisfied.

In order to obtain the solutions of (7), (8), and (9), start with the identities

$$\left\{\begin{array}{l} (a_{11} - \lambda)\delta_{11}(\lambda) + \cdots + a_{1n}\delta_{n1}(\lambda) \underset{\lambda}{\equiv} 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n-1,1}\delta_{11}(\lambda) + \cdots + a_{n-1,n}\delta_{n1}(\lambda) \underset{\lambda}{\equiv} 0, \\ a_{n1}\delta_{11}(\lambda) + \cdots + (a_{nn} - \lambda)\delta_{n1}(\lambda) \underset{\lambda}{\equiv} D(\lambda), \end{array}\right. \tag{10}$$

where $\delta_{11}(\lambda), \cdots, \delta_{n1}(\lambda)$ are the cofactors of the elements $a_{n1}, \cdots, a_{nn} - \lambda$ respectively in $D(\lambda)$. The third subscript of the δ_{j1k}, which indicates the value of λ under consideration, will be omitted except when it is necessary to distinguish notationally between several groups of cofactors of the same order. In writing the identities (10) it has been assumed, for convenience, that the first $n - 1$ rows of $D(\lambda_1)$ contain a non-vanishing first-minor.

On differentiating (10) twice with respect to λ and setting $\lambda = \lambda_1$, it is found that the relations

$$(11) \qquad (a_{ii} - \lambda_1)\delta_{i1} + \sum_{k=1}^{n} a_{ik}\delta_{k1} = 0 \qquad (i = 1, \cdots, n;\ k \neq i),$$

$$(12) \qquad (a_{ii} - \lambda_1)\delta'_{i1} + \sum_{k=1}^{n} a_{ik}\delta'_{k1} = \delta_{i1} \qquad (i = 1, \cdots, n;\ k \neq i),$$

$$(13) \qquad (a_{ii} - \lambda_1)\delta''_{i1} + \sum_{k=1}^{n} a_{ik}\delta''_{k1} = 2\delta'_{i1} \qquad (i = 1, \cdots, n;\ k \neq i),$$

are satisfied. In relations (12) and (13) the notation

$$\delta'_{ik} = \left[\frac{d}{d\lambda}\ \delta_{ik}(\lambda)\right]_{\lambda=\lambda_1}, \qquad \delta''_{ik} = \left[\frac{d^2}{d\lambda^2}\ \delta_{ik}(\lambda)\right]_{\lambda=\lambda_1}, \qquad \delta_{ik}^{(a)} = \left[\frac{d^a}{d\lambda^a}\ \delta_{ik}(\lambda)\right]$$

has been used. Relations (11), (12), and (13) give, at once, the following solutions of equations (7), (8), and (9) respectively:

$$(14) \qquad A_i^{(1)} = c_1\delta_{i1} \qquad (i = 1, \cdots, n),$$

$$(15) \qquad A_i^{(2)} = 2c_1\delta'_{i1} + c_2\delta_{i1} \qquad (i = 1, \cdots, n),$$

$$(16) \qquad A_i^{(3)} = c_1\delta''_{i1} + c_2\delta'_{i1} + c_3\delta_{i1} \qquad (i = 1, \cdots, n),$$

where c_1, c_2, and c_3 are arbitrary constants. On substituting in (13) the values of the $A_i^{(j)}$ just determined, one obtains three solutions of (1) which can be written in the following form:

$$(17) \quad x_i = c_1(\delta''_{i1} + 2\delta'_{i1}t + \delta_{i1}t^2)e^{\lambda_1 t} + c_2(\delta'_{i1} + \delta_{i1}t)e^{\lambda_1 t} + c_3\delta_{i1}e^{\lambda_1 t} \qquad (i = 1, \cdots, n).$$

The remaining $n - 3$ solutions of (1) are given by (6), of Section 2.

CASE III. All δ_{i1k} are zero, at least one δ_{i2k} is not zero, and at least one δ'_{i1k} is different from zero.

By the procedure employed in Case I, one obtains the two purely exponential solutions of (1)

$$(18) \qquad x_i = c_1\delta_{i21}e^{\lambda_1 t} + c_2\delta_{i22}e^{\lambda_1 t} \qquad (i = 1, \cdots, n).$$

The remaining solution of System (1) corresponding to $\lambda = \lambda_1$ is obtained from the first of the three solutions included in (17). For the hypotheses of Case III this solution becomes

$$(19) \qquad x_i = c_3(\delta''_{i1} + 2\delta'_{i1}t)e^{\lambda_1 t} \qquad (i = 1, \cdots, n).$$

The remaining $n - 3$ solutions of (1) are given by (6) of Section 2.

4. *Fundamental Sets of Solutions.* For each of the cases considered in the two preceding articles n solutions of equations (1) were obtained, but it was not shown that they form a fundamental set. Although this property will be established in Section 7 for every case that can arise, yet it seems advisable at this point to illustrate the method by proving the independence of the n solutions in Case II, of Section 3.

The n solutions of the system of differential equations (1) will form a fundamental set if, and only if, their determinant $\Delta(t)$ is different from zero. Suppose that the solutions

$$(20)\qquad \begin{aligned} x_j = {} & c_1(\delta''_{j1} + 2\delta'_{j1}t + \delta_{j1}t^2)e^{\lambda_1 t} + c_2(\delta'_{j1} + \delta_{j1}t)e^{\lambda_1 t} \\ & + c_3\delta_{j1}e^{\lambda_1 t} + \sum_{i=4}^{n} c_i\delta_{j1i}e^{\lambda_i t} \qquad (j = 1, \cdots, n), \end{aligned}$$

do not form a fundamental set, and hence that $\Delta(t) = 0$. One easily verifies that

$$\Delta(t) = \Delta(0)e^{(a_{11}+\cdots+a_{nn})t},$$

where

$$(21)\qquad \Delta(0) = \begin{vmatrix} \delta''_{11}, & \delta'_{11}, & \delta_{11}, & \delta_{114}, & \cdots, & \delta_{11n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \delta''_{n1}, & \delta'_{n1}, & \delta_{n1}, & \delta_{n14}, & \cdots, & \delta_{n1n} \end{vmatrix} = 0,$$

in consequence of the assumption $\Delta(t) = 0$.

The assumption (21) implies that there exist n constants $c_1^{(0)}, \cdots, c_n^{(0)}$ not all of which are zero, such that

$$(22)\qquad c_1^{(0)}\delta''_{j1} + c_2^{(0)}\delta'_{j1} + c_3^{(0)}\delta_{j1} + c_4^{(0)}\delta_{j14} + \cdots + c_n^{(0)}\delta_{j1n} = 0 \qquad (j = 1, \cdots, n).$$

For this special set of values of the c_i, equations (20) become

$$(23)\qquad \begin{aligned} x_j = {} & c_1^{(0)}(\delta''_{j1} + 2\delta'_{j1}t + \delta_{j1}t^2)e^{\lambda_1 t} + c_2^{(0)}(\delta'_{j1} + \delta_{j1}t)e^{\lambda_1 t} \\ & + c_3^{(0)}\delta_{j1}e^{\lambda_1 t} + \sum_{i=4}^{n} c_i^{(0)}\delta_{j1i}e^{\lambda_i t} \qquad (j = 1, \cdots, n). \end{aligned}$$

Thus (23) is a solution of (1) having, from (22), the initial conditions $x_1(0) = x_2(0) = \cdots = x_n(0) = 0$. The trivial solution $x_1 \equiv x_2 \equiv \cdots \equiv x_n \equiv 0$ also has the foregoing initial conditions. In view of the uniqueness of the solutions of (1) under given initial conditions, it follows that the x_j of (23) are identically zero in t.

Take any $x_j \equiv 0$ of (23) and differentiate the identity $n-1$ times with

respect to t. On setting $t = 0$ in each of these n identities, one obtains the n equations

$$(24)\quad \begin{cases} 0\delta''_{j1}c_1^{(0)} + 0\delta'_{j1}c_2^{(0)} + 1\delta_{j1}c_3^{(0)} + 1\delta_{j14}c_4^{(0)} + \cdots + 1\delta_{j1n}c_n^{(0)} = 0, \\ 0\delta''_{j1}c_1^{(0)} + 1\delta'_{j1}c_2^{(0)} + \lambda_1\delta_{j1}c_3^{(0)} + \lambda_4\delta_{j14}c_4^{(0)} + \cdots + \lambda_n\delta_{j1n}c_n^{(0)} = 0, \\ 2\delta''_{j1}c_1^{(0)} + 2\lambda_1\delta'_{j1}c_2^{(0)} + \lambda_1^2\delta_{j1}c_3^{(0)} + \lambda^2{}_4\delta_{j14}c_4^{(0)} + \cdots + \lambda^2{}_n\delta_{j1n}c_n^{(0)} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (n-1)(n-2)\lambda_1^{n-3}\delta''_{j1}c_1^{(0)} + (n-1)\lambda_1^{n-2}\delta'_{j1}c_2^{(0)} + \lambda_1^{n-1}\delta_{j1}c_3^{(0)} \\ \qquad + \lambda_4^{n-1}\delta_{j14}c_4^{(0)} + \cdots + \lambda_n^{n-1}\delta_{j1n}c_n^{(0)} = 0, \end{cases}$$

which are linear and homogeneous in $c_1^{(0)}, \cdots, c_n^{(0)}$. Since not all of the $c_i^{(0)}$ are zero, it follows that the determinant Γ of the coefficients must satisfy the condition

$$(25)\quad \Gamma = \delta''_{j1}\delta'_{j1}\delta_{j1}\delta_{j14}\cdots\delta_{j1n} \begin{vmatrix} 0, & 0, & 1, & \cdots, & 1 \\ 0, & 1, & \lambda_1, & \cdots, & \lambda_n \\ 2, & 2\lambda_1, & \lambda_1^2, & \cdots, & \lambda_n^2 \\ 6\lambda_1, & 3\lambda_1^2, & \lambda_1^3, & \cdots, & \lambda_n^3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (n-1)(n-2)\lambda_1^{n-3}, & (n-1)\lambda_1^{n-2}, & \lambda_1^{n-1}, & \cdots, & \lambda_n^{n-1} \end{vmatrix} =$$

which can be written in the convenient form

$$(26)\qquad \Gamma = \delta''_{j1}\delta'_{j1}\delta_{j1}\delta_{j14}\cdots\delta_{j1n}\Gamma_1 = 0.$$

That the determinant Γ_1 is different from zero is easily shown, for, let $\Delta(\lambda)$ be the determinant obtained by replacing the elements of the first column of Γ_1 by $1, \lambda, \lambda^2, \cdots, \lambda^{n-1}$ respectively. The equation $\Delta(\lambda) = 0$ of degree $n-1$ in λ has $\lambda_1, \lambda_1, \lambda_4, \cdots, \lambda_n$ as its $n-1$ roots. Consequently

$$\left[\frac{d^2\Delta(\lambda)}{d\lambda^2}\right]_{\lambda=\lambda_1} = \Gamma_1 \neq 0,$$

for otherwise $\lambda = \lambda_1$ would be a triple root of $\Delta(\lambda) = 0$. Since $\Gamma_1 \neq 0$ at least one of the factors in (26), multiplying Γ_1, must be zero. On omitting the corresponding column and considering the first $n-1$ equations in (24), one proves similarly that a second of the factors of (26) must be zero. By repeating this procedure one ultimately shows that

$$\delta''_{j1} = \delta'_{j1} = \delta_{j1} = \delta_{j14} = \cdots = \delta_{j1n} = 0 \qquad (j = 1, \cdots, n).$$

This result contradicts the fact that in each of the n solutions of (1) there is at least one non-zero coefficient. Therefore the assumption that $\Delta(t) = 0$

is false, and hence the n solutions of Case II, Section 3, form a fundamental set of solutions of equations (1).

5. *Derivation of a General Form of Solution for System* (1). It will now be shown that when $D(\lambda_1)$ and its minors have certain properties, the system of differential equations

$$x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad (i = 1, \cdots, n), \tag{1}$$

has a solution of the form

$$x_i = (A_i^{(1)}t^\omega + A_i^{(2)}t^{\omega-1} + \cdots + A_i^{(\omega)}t + A_i^{(\omega+1)})e^{\lambda_1 t} \qquad (i = 1, \cdots, n), \tag{2}$$

where the $A_i^{(j)}$ are constants and λ_1 is a root of the characteristic equation $D(\lambda) = 0$. On substituting (2) in (1) and equating coefficients of $t^\omega, \cdots, t^0$ one obtains as sufficient conditions for the existence of a solution of the form (2) the $\omega + 1$ sets of equations

$$(a_{ii} - \lambda_1)A_i^{(1)} + \sum_{k=1}^{n} a_{ik}A_k^{(1)} = 0 \quad (i = 1, \cdots, n; k \neq i), \tag{3}$$

$$(a_{ii} - \lambda_1)A_i^{(2)} + \sum_{k=1}^{n} a_{ik}A_k^{(2)} = \omega A_i^{(1)} \quad (i = 1, \cdots, n; k \neq i), \tag{4}$$

. . . . . .

$$(a_{ii} - \lambda_1)A_i^{(h)} + \sum_{k=1}^{n} a_{ik}A_k^{(h)} = (\omega - h + 2)A_i^{(h-1)} \qquad (i = 1, \cdots, n; k \neq i), \tag{5}$$

. . . . . .

$$(a_{ii} - \lambda_1)A_i^{(\omega+1)} \sum_{k=1}^{n} a_{ik}A_k^{(\omega+1)} = A_i^{(\omega)} \quad (i = 1, \cdots, n; k \neq i), \tag{6}$$

The solutions of these $\omega + 1$ sets of equations will be expressed in terms of the cofactors of $D(\lambda_1)$ and their λ-derivatives, and consequently it will be necessary to consider in detail the determinant

$$D(\lambda) = \begin{vmatrix} a_{11}-\lambda, & a_{12}, & \cdots, & a_{1,n-j}, & a_{1,n-j+1}, & \cdots, & a_{1n} \\ a_{21}, & a_{22}-\lambda, & \cdots, & a_{2,n-j}, & a_{2,n-j+1}, & \cdots, & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-j,1}, & a_{n-j,2}, & \cdots, & a_{n-j,n-j}-\lambda, & a_{n-j,n-j+1}, & \cdots, & a_{n-j,n} \\ a_{n-j+1,1}, & a_{n-j+1,2}, & \cdots, & \cdots, & a_{n-j+1,n-j+1}-\lambda, & \cdots, & a_{n-j+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1}, & a_{n2}, & \cdot & \cdot & \cdot & \cdot, & a_{nn}-\lambda \end{vmatrix}.$$

where the jth and $(j-1)$st principal minors, of orders $n-j$ and $n-j+1$ respectively, have been set off with lines because of the important role that they play in the subsequent theory. On referring to $D(\lambda)$ one readily verifies the identities

$$(7)\quad \begin{cases} (a_{ii}-\lambda)\delta_{ij}(\lambda) + \sum_{k=1}^{n} a_{ik}\delta_{kj}(\lambda) \underset{\lambda}{\equiv} 0 \quad (i=1,\cdots,n-j;\ k\neq i), \\ (a_{ii}-\lambda)\delta_{ij}(\lambda) + \sum_{k=1}^{n} a_{ik}\delta_{kj}(\lambda) \underset{\lambda}{\equiv} \delta_{i,j-1}(\lambda)\ (i=n-j+1,\cdots,n;\ k\neq \end{cases}$$

where the $\delta_{ij}(\lambda)$ $(i=1,\cdots,n-j+1)$ are the jth minor cofactors, respectively (of order $n-j$), of the elements $a_{n-j+1,i}$ $(i=1,\cdots,n-j+1)$ in the upper left $(j-1)$st minor of $D(\lambda)$; and the $\delta_{i,j-1}(\lambda)$ $(i=n-j+1,\cdots,n)$ are the upper left $(j-1)$st minors of $D(\lambda)$ obtained by replacing the $(n-j+1)$st row by the $(n-j+1)$st, $(n-j+2)$nd, $\cdots$, nth rows, respectively.

For the foregoing particular choice in the position of the $(j-1)$st minor cofactors of $D(\lambda)$ occurring in (7) the δ_{kj} $(k=n-j+2,\cdots,n)$ are identically zero, by definition, and have been written for the purpose of securing generality in notation and logic, and also for the purpose of making the coefficients in (7) agree exactly, in form, with the coefficients in the left members of equations (3), $\cdots$, (6). This specialization in the position, in $D(\lambda)$, of the $(j-1)$st minors in question has been employed solely for the purpose of attaining simplicity in presentation; all subsequent work, however, is entirely general and applies to any minor-cofactors of $D(\lambda)$ satisfying certain conditions which will now be specified.

Suppose (a), that $(\lambda-\lambda_1)^s$ is a factor of every $(j-1)$st minor-cofactor (of order $n-j+1$) of $D(\lambda)$, while at least one $(j-1)$st minor-cofactor does not contain $(\lambda-\lambda_1)^{s+1}$; and (b), that $(\lambda-\lambda_1)^r$ is a factor of every jth minor-cofactor of $D(\lambda)$, while at least one jth minor-cofactor does not contain $(\lambda-\lambda_1)^{r+1}$, where obviously $s-r\geqq 1$. Let identities (7) be differentiated $s-1$ times successively with respect to λ, setting $\lambda=\lambda_1$, in each of the s sets of relations thus obtained. By hypothesis (a), the right members of the resulting s sets of relations vanish and, by (b), each term of the left members of the first r sets of equations also vanishes. There remain, however, the following $s-r$ non-trivial sets of relations:

$$(8)\quad (a_{ii}-\lambda_1)\delta_{ij}^{(r)} + \sum_{k=1}^{n} a_{ik}\delta_{kj}^{(r)} = r\delta_{ij}^{(r-1)} = 0\ (i=1,\cdots,n;\ k\neq i),$$

$$(9)\qquad (a_{ii}-\lambda_1)\delta_{ij}^{(r+1)}+\sum_{k=1}^{n}a_{ik}\delta_{kj}^{(r+1)}=(r+1)\delta_{ij}^{(r)} \qquad (i=1,\cdots,n;\ k\neq i),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(10)\qquad (a_{ii}-\lambda_1)\delta_{ij}^{(r+h-1)}+\sum_{k=1}^{n}a_{ik}\delta_{kj}^{(r+h-1)}=(r+h-1)\delta_{kj}^{(r+h-2)} \qquad (i=1,\cdots,n;\ k\neq i),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(11)\qquad (a_{ii}-\lambda_1)\delta_{ij}^{(s-1)}+\sum_{k=1}^{n}a_{ik}\delta_{kj}^{(s-1)}=(s-1)\delta_{ij}^{(s-2)} \qquad (i=1,\cdots,n;\ k\neq i).$$

It is to be remembered that at least $j-1$ of the $\delta_{1j},\cdots,\delta_{nj}$ are identically zero in λ, the remaining $n-j+1$ of the δ_{ij} being the jth minor-cofactors of the elements of some row of a $(j-1)$st minor-cofactor of $D(\lambda_1)$. Furthermore, the choice of the $(j-1)$st and jth minor-cofactors occurring in (7) is restricted only by conditions (a) and (b), where it is to be understood that at least one of the $\delta_{i,j-1}(\lambda)$ does not contain $(\lambda-\lambda_1)^{s+1}$, and at least one of the $\delta_{kj}(\lambda)$ $(k=1,\cdots,n)$ does not contain $(\lambda-\lambda_1)^{r+1}$.

If now the arbitrary ω in (2) be chosen so that $\omega=s-r-1$, it is observed that the $\omega+1$ sets of relations (8), $\cdots$, (11) have the same coefficients in their left members as the $\omega+1$ sets of equations (3), $\cdots$, (6). The $A_1^{(h)},\cdots,A_n^{(h)}$ in the left members of (5) occur in the right members of the $(h+1)$st set; corresponding relations hold in equations (8), $\cdots$, (11). One thus obtains, by comparing the sufficient conditions (3), $\cdots$, (6) with the equations (8), $\cdots$, (11), the following solutions of equations (3), $\cdots$, (6):

$$(12)\qquad A_k^{(1)}=C_{1j}\delta_{kj}^{(r)} \qquad (k=1,\cdots,n),$$

$$(13)\qquad A_k^{(2)}=C_{1j}\frac{\omega}{r+1}\delta_{kj}^{(r+1)}+C_{2j}\delta_{kj}^{(r)} \qquad (k=1,\cdots,n),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(14)\qquad \begin{cases} A_k^{(h)}=C_{1j}\dfrac{\omega(\omega-1)\cdots(\omega-h+2)}{(r+1)(r+2)\cdots(r+h-1)}\delta_{kj}^{(r+h-1)} \\ \qquad\qquad +C_{2j}\dfrac{(\omega-1)\cdots(\omega-h+2)}{(r+1)\cdots(r+h-2)}\delta_{kj}^{(r+h-2)} \\ +\cdots+C_{h-1,j}\dfrac{\omega-h+2}{r+1}\delta_{kj}^{(r+1)}+C_{hj}\delta_{kj}^{(r)} \qquad (k=1,\cdots,n), \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(15)\quad \begin{cases} A_k^{(\omega+1)} = C_{1j}\dfrac{\omega!}{(r+1)(r+2)\cdots(s-1)}\delta_{kj}^{(s-1)} \\ \qquad + C_{2j}\dfrac{(\omega-1)!}{(r+1)\cdots(s-2)}\delta_{kj}^{(s-2)} + \cdots \\ \cdots + C_{\omega+1,j}\dfrac{2!}{(r+1)(r+2)}\delta_{kj}^{(r+2)} + C_{\omega j}\dfrac{1}{r+1}\delta_{kj}^{(r+1)} \\ \qquad (k=1,\cdots,n). \end{cases}$$

On substituting the values of the $A_k^{(i)}$ given by (12), $\cdots$, (15) in (2), ω particular solutions of system (1) are found to be

$$(16)\quad \begin{cases} x_k = C_{1j}e^{\lambda_1 t}[\delta_{kj}^{(r)}t^\omega + \dfrac{\omega}{r+1}\delta_{kj}^{(r+1)}t^{\omega-1} + \dfrac{\omega(\omega-1)}{(r+1)(r+2)}\delta_{kj}^{(r+2)}t^{\omega-2} + \\ \qquad \cdots + \dfrac{\omega!}{(r+1)\cdots(s-1)}\delta_{kj}^{(s-1)}] \\ + C_{2j}e^{\lambda_1 t}[\delta_{kj}^{(r)}t^{\omega-1} + \dfrac{\omega-1}{r+1}\delta_{kj}^{(r+1)}t^{\omega-2} + \cdots + \dfrac{(\omega-1)!}{(r+1)\cdots(s-2)}\delta_{kj}^{(s-2)}] \\ + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \qquad + C_{\omega-1,j}e^{\lambda_1 t}[\delta_{kj}^{(r)}t^2 + \dfrac{2}{r+1}\delta_{kj}^{(r+1)}t + \dfrac{2}{(r+1)(r+2)}\delta_{kj}^{(r+2)}] \\ \qquad + C_{\omega j}e^{\lambda_1 t}[\delta_{kj}^{(r)}t + \dfrac{1}{r+1}\delta_{kj}^{(r+1)}] \quad (k=1,\cdots,n). \end{cases}$$

After replacing the $C_{ij}(s-i)!(\omega-i+1)!$ by $(r+1)\cdots(s-i)c_{ij}$ $(i=1,\cdots,\omega)$, the solutions (16) can be written in the form

$$(17)\quad \begin{cases} x_k = c_{1j}e^{\lambda_1 t}\left[\dfrac{\delta_{kj}^{(s-1)}}{(s-1)!} + \dfrac{\delta_{kj}^{(s-2)}t}{1!(s-2)!} + \cdots + \dfrac{\delta_{kj}^{(s-\omega-1)}t^\omega}{\omega!(s-\omega-1)!}\right] \\ + c_{2j}e^{\lambda_1 t}\left[\dfrac{\delta_{kj}^{(s-2)}}{(s-2)!} + \dfrac{\delta_{kj}^{(s-3)}t}{1!(s-3)!} + \cdots + \dfrac{\delta_{kj}^{(s-\omega-1)}t^{\omega-1}}{(\omega-1)!(s-\omega-1)!}\right] \\ + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \qquad + c_{\omega j}e^{\lambda_1 t}\left[\dfrac{\delta_{kj}^{(s-\omega)}}{(s-\omega)!} + \dfrac{\delta_{kj}^{(s-\omega-1)}t}{(s-\omega-1)!}\right] \qquad (k=1,\cdots,n). \end{cases}$$

The ω particular solutions of (1) given by (17) can now be written in the convenient form

$$(18)\qquad x_k = \sum_{\sigma=1}^{\omega} c_{\sigma j}e^{\lambda_1 t}\sum_{i=\sigma}^{\omega+1}\frac{\delta_{kj}^{(s-i)}t^{i-\sigma}}{(s-i)!(i-\sigma)!} \qquad (k=1,\cdots,n).$$

Thus it has been shown that if, for any root $\lambda = \lambda_1$, of $D(\lambda) = 0$ the $(j-1)$st and jth minor cofactors of $D(\lambda)$ have properties (a) and (b), then $\omega = s - r - 1$ solutions of system (1) given by (17) or (18) can be written at once in terms of the jth minor cofactors of $D(\lambda_1)$ and their λ-derivatives. It remains to be shown that for every multiple root of $D(\lambda) = 0$ the corresponding solutions of system (1), not of the type $x_k = A_k e^{\lambda_1 t}$ $(k = 1, \cdots, n)$, can always be obtained directly from form (17) or (18).

6. *Solutions of System* (1) *Corresponding to a Multiple Root of* $D(\lambda) = 0$. Suppose (A) that $\lambda = \lambda_1$ is a root of $D(\lambda) = 0$ of order S_0, and (B) that all 1st, 2nd, $\cdots$, $(q-1)$st minors of $D(\lambda_1)$ are zero, $(q \leqq S_0)$, while at least one qth minor of $D(\lambda_1)$ is different from zero. In consequence of (A) and (B), q purely exponential solutions of (1) are readily obtained in terms of qth minor cofactors of $D(\lambda_1)$. For, consider again the system of differential equations

$$x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad (i = 1, \cdots, n), \tag{1}$$

for which, if possible, solutions of the form

$$x_i = A_i e^{\lambda_1 t} \qquad (i \quad 1, \cdots, n), \tag{2}$$

are to be found. On substituting (2) in (1) there results as sufficient conditions that (2) shall be a solution of (1)

$$(a_{ii} - \lambda_1)A_i + \sum_{k=1}^{n} a_{ik}A_k = 0 \qquad (i = 1, \cdots, n;\ k \neq i), \tag{3}$$

In view of hypothesis (A) and (B) the general solution of equations (3) can be written in the form

$$A_i = c_1\delta_{iq1} + c_2\delta_{iq2} + \cdots + c_q\delta_{iqq} \qquad (i = 1, \cdots, n), \tag{4}$$

where the δ_{iqk} are the qth minor cofactors of $D(\lambda_1)$ obtained in solving (3) by determinants, and the c_j are arbitrary constants. At least $q-1$ of the A_i in (4) will have single-term values of the form $A_i = c_k\delta_{iqk}$. Substitute the A_i given by (4) in (2) and one obtains the q purely exponential solutions of system (1)

$$x_k = c_1\delta_{kq1}e^{\lambda_1 t} + c_2\delta_{kq2}e^{\lambda_1 t} + \cdots + c_q\delta_{kqq}e^{\lambda_1 t} \qquad (k = 1, \cdots, n). \tag{5}$$

In consequence of assumption (B) all derivatives with respect to λ, up to and including the $(q-j-1)$st, of all jth minor cofactors of $D(\lambda)$ vanish

when $\lambda = \lambda_1$, for $j = 1, \cdots, (q-1)$. This follows from the fact that the λ-derivative of every $(j-1)$st minor of $D(\lambda)$ is a sum of jth minors of $D(\lambda)$. Thus hypotheses (B) imply that $(\lambda - \lambda_1)^{q-j}$ is a factor of every jth minor of $D(\lambda)$ $[j = 1, 2, \cdots, (q-1)]$. If there exists at least one first-minor of $D(\lambda)$ which does not contain $(\lambda - \lambda_1)^q$ as a factor, then the remaining $S_0 - q$ solutions of system (1), associated with $\lambda = \lambda_1$, can be written at once by (17) or (18) of Section 5, for the hypotheses of Section 5 are satisfied if

$$S = S_0, \quad \omega = S_0 - q, \quad r = q - 1, \quad j = 1.$$

It may happen, however, that $(\lambda - \lambda_1)^\rho$, where $\rho \geqq q$, is a common factor of all first minors of $D(\lambda)$ in which case there are fewer than $S_0 - q$ solutions of the form (17), of Section 5, which are expressed in terms of the first minor cofactors and their λ-derivatives. In fact, a similar special situation may exist for the jth minors of $D(\lambda)$ for $j = 2, \cdots, (q-1)$. The assumptions with respect to this question will therefore be generalized so as to include every possible special case.

Hence let it be assumed (C) that every jth minor of $D(\lambda)$ has the factor $(\lambda - \lambda_1)^{S_j}$, while at least one jth minor does not contain $(\lambda - \lambda_1)^{S_j+1}$ $(j = 1, 2, \cdots, q)$, where obviously $S_j \geqq q - j$ and $S_{j-1} - S_j \geqq 1$. Hypotheses (C) are identical in character with the assumptions underlying the derivation of forms (17) and (18), of Section 5, and consequently the results of the preceding article are immediately applicable. The solutions of equations (1) thus obtained by successive application of forms (17) and (18) of Section 5, are, in general, expressed in terms of 1st, 2nd, $\cdots$, qth minor cofactors of $D(\lambda_1)$ and their λ derivatives. In obtaining solutions in terms of jth minor-cofactors of $D(\lambda)$, $(j = 1, \cdots, q)$, one makes the following substitutions in (17) or (18), of Section 5:

$$(6) \qquad s = S_{j-1}, \quad \omega = S_{j-1} - S_j - 1 = \omega_j, \quad r = S_j.$$

The total number of solutions, corresponding to $\lambda = \lambda_1$ which one obtains in this way, is clearly given by

$$\omega_1 + \omega_2 + \cdots + \omega_q = (S_0 - S_1 - 1) + (S_1 - S_2 - 1) + \cdots + (S_{q-1} - S_q - 1) = S_0 - S_q - q = S_0 - q,$$

since, by (B), $S_q = 0$. There are also q solutions depending on the root $\lambda = \lambda_1$, which are given by equations (5). Therefore there are S_0 solutions

of System (1) associated with $\lambda = \lambda_1$ which is a root of $D(\lambda) = 0$ of multiplicity S_0. They are

$$(7)\quad x_k = \sum_{j=1}^{q} \sum_{\sigma=1}^{\omega} c_{\sigma j} e^{\lambda_1 t} \sum_{i=\sigma}^{\omega+1} \frac{\delta_{kj}^{(s-i)} t^{i-\sigma}}{(s-i)\,!\,(i-\sigma)\,!} + \sum_{i=1}^{q} c_i \delta_{kqi} e^{\lambda_1 t} \qquad (k = 1, \cdots, n).$$

The solutions corresponding to every other root of $D(\lambda) = 0$ can be found similarly.

7. *Independence of the n Solutions of System* (1). The independence of the n solutions of (1) will be established by proving that their determinant $\Delta(t)$ is distinct from zero. In order to attain complete generality in the following proof, assume that the characteristic equations $D(\lambda) = 0$ has β roots, $\lambda_1, \lambda_2, \cdots, \lambda_\beta$ of orders $e_1, e_2, \cdots, e_\beta$, respectively, where clearly $\beta \leqq n$ and $e_1 + e_2 + \cdots + e_\beta = n$. Let

$$(2)\qquad x_1 = \phi_{1i}, \cdots, x_n = \phi_{ni} \qquad (i = 1, \cdots, n),$$

represent the n solutions of the system

$$(1)\qquad x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad (i = 1, \cdots, n).$$

The determinant of the n solutions is

$$(3)\qquad \Delta(t) \equiv |\,\phi_{ij}\,| = \Delta(0)\, e^{(a_{11} + \cdots + a_{nn})t}.$$

The indirect method of proof will be employed. Assume

$$(4)\qquad \Delta(t) = 0,$$

which implies the existence of n constants $c_1^{(0)}, \cdots, c_n^{(0)}$, not all of which are zero, such that

$$(5)\qquad c_1^{(0)}\phi_{i1}(0) + \cdots + c_n^{(0)}\phi_{in}(0) = 0 \qquad (i = 1, \cdots, n).$$

From (5) it is evident that

$$(6)\qquad x_i = c_1^{(0)}\phi_{i1} + \cdots + c_n^{(0)}\phi_{in} \qquad (i = 1, \cdots, n),$$

is a solution of system (1) having as initial conditions

$$(7)\qquad x_1(0) = x_2(0) = \cdots = x_n(0) = 0.$$

But $x_1 \equiv x_2 \equiv \cdots \equiv x_n \equiv 0$ is also a solution of (1) having initial condi-

tions (7). In view of the uniqueness of solutions (2) under given initial conditions, (6) becomes

$$(8) \qquad x_i = c_1^{(0)}\phi_{i1} + \cdots + c_n^{(0)}\phi_{in} \underset{t}{\equiv} 0 \qquad (i = 1, \cdots, n).$$

Any x_j of (8) can be written in the form

$$(9) \quad x_j = \sum_{i=0}^{n_1-1} c_i^{(1)} f_{1i}(t) e^{\lambda_1 t} + \cdots + \sum_{i=0}^{n_\beta-1} c_i^{(\beta)} f_{\beta i}(t) e^{\lambda_\beta t} \underset{t}{\equiv} 0,$$

where the $f_{1i}(t), \cdots, f_{\beta i}(t)$ are polynomials in t whose respective degrees are indicated by their second subscripts, while the coefficients occurring in these polynomials are either derivatives of cofactors in $D(\lambda_i)$ or linear combinations of such derivatives. The $n_1 - 1$, $n_2 - 1$, $\cdots$, $n_\beta - 1$ are the highest degrees of the β sets of polynomials $f_{1i}(t)$, $f_{2i}(t), \cdots, f_{\beta i}(t)$ respectively.

Set $n_1 + n_2 + \cdots + n_\beta = p$ and differentiate (9) $p - 1$ times with respect to t, setting $t = 0$ in each of the resulting p relations. One thus obtains the p linear homogeneous relations among the p constants $c_0^{(1)}, \cdots, c^{(1)}{}_{n_1-1}, \cdots, c_0^{(\beta)}, \cdots, c^{(\beta)}{}_{n_\beta-1}$,

$$(10) \qquad x_j(0) = 0, \quad x'_j(0) = 0, \cdots, x_j^{(p-1)}(0) = 0.$$

Since (10) are consistent the determinant of the coefficients of the $c_i^{(k)}$ is zero, or

$$(11) \qquad \Gamma = \prod_{i=1}^{p} \eta_{ji}\, \Gamma_1(\lambda_1, \cdots, \lambda_\beta) = 0,$$

where η_{ji} is the factor common to the elements of the ith column in Γ, and is either a single term of the form $\delta^{(a_i)}_{j\mu_i\nu_i}(\lambda_i)$ or a linear combination of terms of that kind. In accordance with the foregoing notation $\delta^{(a_i)}_{j\mu_i\nu_i}(\lambda_i)$ indicates the a_ith λ-derivative for $\lambda = \lambda_1$ of a μ_ith minor-cofactor in $D(\lambda)$. After removing the factors $\eta_{j1}, \cdots, \eta_{jp}$ from Γ, there remains the determinant

$$(12) \quad \Gamma_1 \equiv \begin{vmatrix} (1)^{(n_1-1)}, & \cdots, & (1)^{(1)}, & 1, & \cdots, & (1)^{(n_\beta-1)}, & \cdots, & (1)^{(1)}, & 1 \\ (\lambda_1)^{(n_1-1)}, & \cdots, & (\lambda_1)^{(1)}, & \lambda_1, & \cdots, & (\lambda_\beta)^{(n_\beta-1)}, & \cdots, & (\lambda_\beta)^{(1)}, & \lambda_\beta \\ (\lambda_1{}^2)^{(n_1-1)}, & \cdots, & (\lambda_1{}^2)^{(1)}, & \lambda_1{}^2, & \cdots, & (\lambda_\beta{}^2)^{(n_\beta-1)}, & \cdots, & (\lambda_\beta{}^2)^{(1)}, & \lambda_\beta{}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (\lambda_1{}^{p-1})^{(n_1-1)}, & \cdots, & (\lambda_1{}^{p-1})^{(1)}, & \lambda_1{}^{p-1}, & \cdots, & (\lambda_\beta{}^{p-1})^{(n_\beta-1)}, & \cdots, & (\lambda_\beta{}^{p-1})^{(1)}, & \lambda_\beta{}^{p-1} \end{vmatrix}$$

where for convenience in writing

$$(\lambda_i{}^j)^{(k)} = \left[\frac{d^k \lambda^j}{d\lambda^k}\right]_{\lambda=\lambda_i} .$$

In order to show that Γ_1 is not zero, replace the elements of the first column of Γ_1 by $1, \lambda, \lambda^2, \cdots, \lambda^{p-1}$ respectively, and designate the resulting determinant by $\Delta(\lambda)$. The equation $\Delta(\lambda) = 0$ is of degree $p-1$ and has the roots $\lambda_1, \lambda_2, \cdots, \lambda_s$ of orders $n_1 - 1, n_2, n_3, \cdots, n_s$ respectively. Since $n_1 - 1 + n_2 + \cdots + n_s = p - 1$, all of the roots of $\Delta(\lambda) = 0$ are accounted for. It is thus evident that

$$\left[\frac{d^{n_1-1}\Delta(\lambda)}{d\lambda^{n_1-1}}\right]_{\lambda=\lambda_1} = \Gamma_1 \neq 0,$$

and consequently some η_{ji} in (11), say η_{j1}, is zero. On omitting the last equation in (10) and suppressing the terms of the first column of Γ, since $\eta_{j1} = 0$, it is seen by a repetition of the foregoing argument that a second η_{ji}, say η_{j2}, must also be zero. By this procedure, one ultimately proves that

$$\eta_{j1} = \eta_{j2} = \cdots = \eta_{jp} = 0. \tag{13}$$

since x_j was any one of the variables $x_1, \cdots, x_n$, it is thus evident that under the assumption $\Delta(t) = 0$ one has (13) for $j = 1, \cdots, n$.

In Section 6 it was seen that for any $\lambda = \lambda_k$ the corresponding solutions of system (1) in terms of jth minor cofactors of $D(\lambda_k)$ have zero values for at least $j-1$ of the variables $x_1, \cdots, x_n$. In view of this property it follows that for each λ_i there is an x_j for which at least one of the corresponding η_{ji} of (11) is a non-zero term of the form $\delta^{(a_i)}_{j\mu_i\nu_i}(\lambda_i)$. Hence (13) contradicts the established properties of the solutions of system (1) and therefore the assumption $\Delta(t) = 0$, on which the derivation of (13) was based, is erroneous. This establishes the fact that the n solutions of equations (1) given in Section 6 form a fundamental set.

8. *Systems of Non-Homogeneous Differential Equations.* Consider the system of non-homogeneous equations

$$x'_i = a_{i1}x_1 + \cdots + a_{in}x_n + \theta_i \quad (i = 1, \cdots, n), \tag{1'}$$

which differ from the system of homogeneous equations

$$x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \quad (i = 1, \cdots, n), \tag{1}$$

by the $\theta_i(t)$ which are assumed to be analytic functions of t. Let

$$x_1 = \phi_{1i}(t), \cdots, x_n = \phi_{ni}(t) \quad (i = 1, \cdots, n), \tag{2}$$

represent a fundamental set of solutions of (1). Change from the dependent

variables x_i to new dependent variables y_i by the transformation

$$x_i = \phi_{i1}y_1 + \cdots + \phi_{in}y_n \qquad (i = 1, \cdots, n). \tag{3}$$

Since the ϕ_{ij} satisfy equations (1), the result of substituting (3) in (1) is

$$y'_1\phi_{i1} + \cdots + y'_n\phi_{in} = C_i \qquad (i = 1, \cdots, n). \tag{4}$$

On solving (4) for the y_i, one has

$$\left\{\begin{aligned} y'_i &= \frac{1}{\Delta(t)} \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_{j1}, & \cdots, & \phi_{j,i-1}, & \theta_j, & \phi_{j,i+1}, & \cdots, & \phi_{jn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \\ &= \frac{e^{\sum_{k=1}^{n}\lambda_k t - \lambda_i t}}{\Delta(0)e^{\sum_{k=1}^{n}\lambda_k t}} \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{j1}, & \cdots, & p_{j,i-1}, & \theta_j, & p_{j,i+1}, & \cdots, & p_{jn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \\ &= \frac{e^{-\lambda_i t}}{\Delta(0)} \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{j1}, & \cdots, & p_{j,i-1}, & \theta_j, & p_{j,i-1}, & \cdots, & p_{jn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \end{aligned}\right. \qquad (i = 1, \cdots, n), \tag{5}$$

where $e^{\lambda_j t}p_{ij} \equiv \phi_{ij} \qquad (i, j = 1, \cdots, n).$

On expanding the determinants occurring in (5), one can write the y'_i in the form

$$y'_i = \sum_{j=1}^{n} \theta_j P_{ji} e^{-\lambda_i t} \qquad (i = 1, \cdots, n), \tag{6}$$

where the $P_{ji}\Delta(0)$ is the cofactor of p_{ji} in the determinant $| p_{ji} |$. Since t is the only variable occurring in the right members of (6), the y_i are obtained by quadratures, and hence

$$y_i = \sum_{j=1}^{n} \int \theta_j P_{ji} e^{-\lambda_i t} dt + c_i \qquad (i = 1, \cdots, n). \tag{7}$$

On substituting (7) in (3), one has as the general solution of equations (1)

$$x_i = \sum_{k=1}^{n} e^{\lambda_k t} p_{ik} \left\{ \sum_{j=1}^{n} \int \theta_j P_{jk} e^{-\lambda_k t} dt \right\} + \sum_{j=1}^{n} c_j \phi_{ij} \qquad (i = 1, \cdots, n). \tag{8}$$

A brief analysis of the nature of the solutions (8) will be given for each of the following important cases:

CASE I. The characteristic equation $D(\lambda) = 0$ has no multiple root and (a), the θ_i are polynomials in t; (b) the θ_i are linear functions of Sin at and Cos at; and (c) the θ_i are linear in Sin at and Cosat and polynomials in t.

CASE II. The characteristics equation $D(\lambda) = 0$ has one or more multiple roots which give rise to solutions involving polynomials in t, and the θ_i have the properties (a), (b), and (c) as in Case I.

Case I(a). For this case the P_{jk} are constants and therefore the general solution (8) becomes

$$(9) \qquad x_i = \sum_{j,k=1}^{n} p_{ik} P_{jk} e^{\lambda_k t} \int \theta_j e^{-\lambda_k t} dt + \sum_{j=1}^{n} c_j \phi_{ij} \quad (i = 1, \cdots, n).$$

After the indicated integrations are performed (9) can be written in the form

$$(10) \qquad x_i = \psi_i(t) + \sum_{j=1}^{n} c_j \phi_{ij} \quad (i = 1, \cdots, n).$$

where the ψ_i are polynomials in t whose degrees do not exceed the highest degree of t in the θ_i.

Case I(b). The quadratures occurring in (8) are now of the form

$$\int \begin{pmatrix} \text{Sin } at \\ \text{Cos } at \end{pmatrix} e^{-\lambda_k t} dt.$$

If $a^2 + \lambda_k^2 \neq 0$ $(k = 1, \cdots, n)$, then on performing the integrations indicated in (8) one obtains as the general solution of (1′)

$$(11) \qquad x_i = R_i (\text{Sin } at, \text{Cos } at) + \sum_{j=1}^{n} c_j \phi_{ij} \quad (i = 1, \cdots, n),$$

where the R_i are linear in Sin at and Cos at. But if $a^2 + \lambda_k^2 = 0$, then for $\lambda_k = +a\sqrt{-1}$,

$$\int \begin{pmatrix} \text{Sin } at \\ \text{Cos } at \end{pmatrix} e^{-\lambda_k t} dt = \tfrac{1}{2} \begin{pmatrix} -\sqrt{-1} \\ 1 \end{pmatrix} \left[t \pm \frac{1}{2a\sqrt{-1}} e^{-2at\sqrt{-1}} \right] .$$

For $\lambda_k = -a\sqrt{-1}$, the corresponding value of the preceding integral is obtained by replacing a by $-a$. The solution of (1′) now takes the form

$$(12) \qquad x_i = R_i(\text{Sin } at, \text{Cos } at) + h_i t(\text{Cos } at \pm i \text{ Sin } at) + \sum_{j=1}^{n} c_j \phi_{ij} \quad (i = 1, \cdots, n),$$

where the R_i are of the same form as in (11) and the h_i are constants.

Case I(c). For this case the quadratures of (8) are of the type

$$\int \begin{pmatrix} \text{Sin } at \\ \text{Cos } at \end{pmatrix} t^m e^{-\lambda_k t} dt.$$

After performing these integrations the general solution of (1′) is

$$(13) \qquad x_i = H_i(t, \operatorname{Sin} at, \operatorname{Cos} at) + \sum_{j=1}^{n} c_j \phi_{ij} \quad (i = 1, \cdots, n),$$

where the Sin at and Cos at enter linearly but have coefficients that are either constants or polynomials in t of the same degree as the θ_i. The special case for which $\lambda_k = \pm a\sqrt{-1}$ makes no change in the form of solution (13) except that the degree of the polynomials in t is increased by unity.

CASE II. For each of the cases (a), (b), and (c), the P_{jk} of (8) are polynomials in t. In II(a) the integrals are of the same type as those in I(a) and consequently the solution of (1′) for this case is of form (10) where, in general, the degrees of the ψ_i exceed the maximum degree of the θ_i. Cases II(b) and II(c) lead to quadratures of the same type as those of I(c) and hence the solution of (1′) in either of these two cases can be written in the form (13).

UNIVERSITY OF CHICAGO,
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On Generalizations of the Bernoullian Functions and Numbers.

By E. T. Bell.

1. Both the numbers and the functions of Bernoulli can be generalized in an infinity of ways. Many of these generalizations have either an interest of their own or an importance in analysis. The *methods* by which generalizations are attainable can conveniently be segregated into two classes. Into the first fall all methods dependent upon modifications of the difference equations defining the Bernoullian functions. The recent generalizations of Nörlund * belong to the first class. As we shall not again refer to this class it is unnecessary to describe it further. Into the second fall all methods dependent upon modifications of the symbolic equations of definition of the Bernoullian functions. The symbolic calculus implied is that of Blissard (and independently also of Lucas), in which the symbol A is the umbra, or representative, of the class of numbers or functions A_n $(n = 0, 1, 2, \cdots)$, and A_n is written A^n until after the completion of all ordinary algebraic processes, when the exponents of umbrae are degraded to suffixes. During the intermediate steps powers A^n, B^n, $\cdots$ of umbrae A, B, $\cdots$ are treated as ordinary algebraic quantities, or ordinaries. For example, the first derivative of A^n with respect to A is nA^{n-1}, umbrally, and this is nA_{n-1} finally. We shall illustrate the second class of generalizations by an interesting example, which itself generalizes an extension of the Bernoullian functions and numbers due to Krause and Appell.

2. As the substance of Blissard's long neglected papers is now well known, it will be sufficient here to recall a few essentials of his calculus as we shall use it.

A function $f(x^{(1)}, x^{(2)}, \cdots, x^{(k)})$ in which each $x^{(i)}$ $(i = 1, \cdots, k)$ is replaced by the umbra A is evaluated as follows: the $x^{(i)}$ are replaced by k *different* letters A, B, $\cdots$, C, each of which denotes an umbra, and *after* the degradation of exponents each of A, B, $\cdots$, C is replaced by A. For example,

$$(A + Aa)^1 \equiv (A + Ba)^1 = A^1B^0a^0 + A^0B^1a^1 = A_1B_0 + A_0B_1a,$$

* "Mémoire sur les polynomes de Bernoulli," *Acta Mathematica*, 43 (1922), pp. 121-196.

if a is an ordinary, and the final value is therefore $A_0A_1(1+a)$. Note particularly that zeroth powers of umbrae must always be indicated explicitly. In particular, $A, B, \cdots, C$ being umbrae, $a, b, \cdots, c$ ordinaries,

$$(aA + bB + \cdots + cC)^0 = A_0B_0 \cdots C_0;$$

and the umbral multinomial theorem is, for $n \geqq 0$ an integer (this theorem can be extended to negative and fractional exponents),

$$(aA + bB + \cdots + cC)^n = \frac{n!}{\alpha!\,\beta! \cdots \gamma!}\, a^\alpha b^\beta \cdots c^\gamma\, A_\alpha B_\beta \cdots C_\gamma,$$

the $\sum$ extending to all sets $(\alpha, \beta, \cdots, \gamma)$ of integers $\alpha, \beta, \cdots, \gamma$ such that

$$0 \leqq \alpha, \beta, \cdots, \gamma; \qquad \alpha + \beta + \cdots + \gamma = n.$$

The umbral exponential theorem is, θ being a parametric ordinary, A an umbra,

$$e^{A\theta} = \sum_{n=0}^{\infty} A^n\theta^n/n! = \sum A_n\theta^n/n! \quad (0! = 1).$$

This can be considered as an identity of definition. It follows from this definition that we do *not* have, for A an *arbitrary* umbra, $e^{-A\theta}$ equal to the reciprocal of $e^{A\theta}$. The value of $e^{-A\theta}$ is obtained from that of $e^{A\theta}$ by changing the sign of the parameter θ in $e^{A\theta}$:

$$e^{-A\theta} = \sum_{n=0}^{\infty} (-1)^n A_n\theta^n/n!.$$

The one exception in which $e^{-A\theta}$ is the reciprocal of $e^{A\theta}$ is the trivial case in which $A_n = 1$ $(n = 0, 1, 2, \cdots)$.

Infinite series, such as those just discussed, are used merely as generating functions, or *generators*. Thus $e^{A\theta}$ is the generator of $A_n/n!$, this being the coefficient of θ^n. The question of convergence of generators can be disregarded. If this is not obvious, it has been proved in a former paper.* In particular we can equate coefficients of like powers of θ in identities between generators derived by means of the umbral calculus.

It is desirable to have no umbral derivatives in the final statements of results. Such derivatives can always be eliminated by repeated applications of the immediately verifiable identity

$$\frac{\partial^k}{\partial A^k}(\theta + A)^n = \frac{\partial^k}{\partial\theta^k}(\theta + A)^n, .$$

in which θ is an ordinary and A an umbra.

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* *Transactions of the American Mathematical Society*, Vol. 25, pp. 135-154.

Addition of umbrae $v, p, \cdots, t$ is used here only in the following sense. If u is the sum $v + p + \cdots + t$ of $v, p, \cdots, t$, this is written $u = v + p + \cdots + t$, and means that for all integers $n \geqq 0$,

$$u^n \equiv u_n = (v + p + \cdots + t)^n,$$

the last of which is to be expanded by the umbral multinomial theorem. Thus if $u = v + p$, then $u_1 = v_1 p_0 + v_0 p_1, u_2 = v_2 p_0 + 2v_1 p_1 + v_0 p_2$, etc.

3. *Henceforth each of the k letters A, B, · · ·, C represents an umbra of the Bernoulli numbers.*

$$D_0 = 1, \quad D_1 = -\tfrac{1}{2}, \quad D_2 = \tfrac{1}{6}, \quad D_4 = -1/30, \cdots, D_{2n+1} = 0 \quad (n > 0),$$
$$D = A, B, \cdots, C,$$

and each of the k letters $X, Y, \cdots, Z$ represents an umbra of the common Bernoullian functions with the respective ordinary arguments $x, y, \cdots, z$, so that

$$nW_{n-1} \equiv nW_{n-1}(w) = \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} B_r w^{n-r} \quad (n \geqq 0),$$

$$(W, w) \equiv (X, x), \quad (Y, y), \cdots, \quad (Z, z).$$

It is well known (or follows as the simplest case, $k = 1$, of the numbers Ω and functions Λ later discussed in this paper) that the symbolic equations of definition for D, W are

$$(1) \qquad \frac{\theta}{e^\theta - 1} = e^{D\theta}, \quad \frac{e^\theta(1 - e^{w\theta})}{1 - e^\theta} = e^{W\theta}.$$

The arguments $w = x, y, \cdots, z$ are not assumed to be integers unless so stated.

We note first that D_n, W_n are functions of the integer variable n. A generalization which does not seem to have been considered is that in which D, W are functions of p integer variables $\xi, \eta, \cdots, \zeta$,

$$D_{\xi, \eta, \ldots, \zeta}, \qquad W_{\xi, \eta, \ldots, \zeta};$$

that is, these D, W are members of p-fold sequences. A discussion of the case $p = 1$ is a necessary preliminary for the case $p > 1$. We shall therefore attend exclusively to $p = 1$, reserving the similarly treated case $p > 1$ for another occasion.

There is a second possible generalization, applicable to $p \geqq 1$, which we shall pass over here. We have postulated that the arguments $x, y, \cdots, z$

shall be ordinaries. The case in which $x, y, \cdots, z$ are umbral, so that $x^n = x_n$, etc., is treated with equal facility by means of the theory developed in a former paper.*

A third prolific generalization, immediately applicable to all that precedes, consists in removing the restriction that F, introduced presently, shall be algebraic in its arguments, $a, b, x', y', \cdots, z'$. It is now sufficiently evident that the number of generalizations which can be systematically developed is infinite. The possibilities thus suggested will be more readily appreciated after the short treatment of the special case to which this paper is devoted.

4. A power series $\sum_0^\infty t_n\theta^n$ in θ can be written in the umbral form $e^{u\theta}$, where $u_n = n!\, t_n$. Let $F \equiv F(a, b, x', y', \cdots, z')$ be a rational algebraic function of its ordinary arguments $a, b, x', y', \cdots, z'$, and in F make the substitution

$$(a, b, x', y', \cdots, z') \equiv (\theta, e^\theta, e^{x\theta}, e^{y\theta}, \cdots, e^{z\theta}),$$

so that F becomes a function of θ, say $f(\theta)$. Write the power series expansion of $f(\theta)$ in the form $e^{u\theta}$. Then u is an umbra of a system of polynomials in the k variables $x, y, \cdots, z$, and $u^n \equiv u_n = u_n\ (x, y, \cdots, z)$ is the nth such polynomial. For the special values $(x, y, \cdots, z) \equiv (x_0, y_0, \cdots, z_0)$, where $x_0, y_0, \cdots, z_0$ are given constants, the polynomials u become a system of numbers β. We shall say that the numbers β are *associated* with the polynomials u, and that they have the *characteristic* determined by $(x_0, y_0, \cdots, z_0)$. Evidently u is uniquely known when F is given, and likewise for β when in addition the characteristic is assigned.

For simplicity let us restrict the discussion to the case in which the coefficients of F are rational, and the domain (of reducibility) is the rational domain. Then the resolution of the numerator of F into irreducible polynomials in $a, b, x', y', \cdots, z'$ is unique, and likewise for the denominator. Suppose common factors to have been removed. This unique form of F is expanded in the form $e^{u\theta}$, and u is now called a *primary* umbra. From the definitions, primary umbrae arise only from irreducible rational algebraic functions F. The numbers associated with u are called primary. Non-primaries, either polynomials or their respective associated numbers, are called derived.

A further generalization, easily attainable and readily developed by the symbolic method, at once suggests itself. In the foregoing we have replaced $x', y', \cdots, z'$ by $e^x, e^y, \cdots, e^z$ respectively. But we might also replace

* *Mathematische Zeitschrift*, Bd. 19, S. 35.

$x', y', \cdots, z'$ by $e^{P(x)}, e^{Q(y)}, \cdots, e^{R(z)}$, where $P, Q, \cdots, R$ are single valued functions. The case where $P, Q, \cdots, R$ are algebraic is of special interest, because it leads to an immediate generalization of the Bessel coefficients.

It is now evident that we can set up a simple isomorphism between the multiplicative properties of rational algebraic functions in $k+2$ variables $a, b, x, y, \cdots, z$ and the additive properties of umbrae of polynomials in k variables $x, y, \cdots, z$, and similarly for umbrae of the respective associated numbers.

Suppose first that F is irreducible. Let $e^{u\theta}$ be the umbral exponential obtained from F as above outlined. Then from F^n, $n>0$ an integer, is derived $e^{n.u\theta}$, *where* $n.u$ *is a convenient symbol for the sum* $u+v+\cdots+t$ *of* n *umbrae, each of which is to be replaced by* u *after the degradation of exponents.* Thus

$$(3.u)^2 = (u+v+t)^2 = u_2v_0t_0 + u_0v_2t_0 + u_0v_0t_2 + 2u_1v_1t_0 + 2u_1v_0t_1 + 2u_0v_1t_1.$$

If F is reducible, say $F = G^\alpha H^\beta \cdots K^\gamma$, let the respective umbral exponentials of $F, G, H, \cdots, K$ be $e^{u\theta}, e^{v\theta}, e^{t\theta}, \cdots, e^{p\theta}$, where $u, v, t, \cdots, p$ are umbrae. Then clearly since for u', v' arbitrary umbrae, $e^{u'\theta}e^{v'\theta} = e^{(u'+v')\theta}$, we have

$$u = \alpha.v + \beta.t + \cdots + \gamma.p.$$

With reference to umbrae, it follows (cf. § 6), that the rational algebraic functions giving rise to exponentials involving those umbrae play a part analogous to that of antilogarithms in ordinary multiplication. There is also a complete correspondence for division, but we shall not discuss this here.

From the above sketch we see that the algebra of derived umbrae can be inferred from that of the primaries. The common Bernoullian functions X defined in (1) are primaries, since in this case F is $F(e^\theta, e^{\theta x})$, and $F(a, b, x', y', \cdots, z') \equiv F(b, x') = b(1-x')/(1-b)$, which is irreducible in the sense defined.

5. To illustrate the foregoing we take the immediate extension suggested by $F(b, x')$, of § 4 and multiply together k such rational functions. The k letters $a, b, \cdots, c$ denote ordinaries; $x, y, \cdots, z$ are k independent variables, and as in § 3 each of the k umbrae $A, B, \cdots, C$ is an umbra of the Bernoulli numbers.

The Functions Λ,

$$\Lambda^n \equiv \Lambda_n \equiv \qquad \Lambda_n^{(a,\, b,\, \ldots,\, c)}\ (x,\, y,\, \cdots,\, z)$$

are defined by

$$(2)\quad e^{\Lambda\theta} = \frac{e^{a\theta}(1-e^{ax\theta})}{1-e^{a\theta}} \cdot \frac{e^{b\theta}(1-e^{by\theta})}{1-e^{b\theta}} \cdots \frac{e^{c\theta}(1-e^{cz\theta})}{1-e^{c\theta}} .$$

With Λ it is evidently convenient to introduce the following, suggested by the right of (2): the functions of Ψ,

$$\Psi^n \equiv \Psi_n \equiv \Psi_n^{(a,\, b,\, \ldots,\, c)}\ (x,\, y,\, \cdots,\, z),$$

defined by

$$(3)\qquad e^{\Psi\theta} = (1-e^{ax\theta})(1-e^{by\theta}) \cdots (1-e^{cz\theta});$$

the numbers Ω,

$$\Omega^n \equiv \Omega_n \equiv \Omega_n^{(a,\, b,\, \ldots,\, c)},$$

whose definition is

$$(4)\qquad e^{\Omega\theta} = \frac{a\theta}{e^{a\theta}-1} \cdots \frac{b\theta}{e^{b\theta}-1} \cdots \frac{c\theta}{e^{c\theta}-1} ;$$

the numbers Θ,

$$\Theta^n \equiv \Theta_n \equiv \Theta_n^{(a,\, b,\, \ldots,\, c)},$$

defined by

$$(5)\qquad e^{\Theta\theta} = (1-e^{a\theta})(1-e^{b\theta}) \cdots (1-e^{c\theta}),$$

and the functions Φ,

$$\Phi^n \equiv \Phi_n \equiv \Phi_n^{(a,\, b,\, \ldots,\, c)}\ (x,\, y,\, \cdots,\, z),$$

whose definition is

$$(6)\qquad e^{\Phi\theta} = \frac{ax\theta}{e^{ax\theta}-1} \cdot \frac{by\theta}{e^{by\theta}-1} \cdots \frac{cz\theta}{e^{cz\theta}-1} .$$

Each of Λ_n, Ψ_n, Ω_n, Θ_n, Φ_n, is of *rank n, order k* (= the number of independent ordinary variables $x, y, \cdots, z$, or the number of ordinary parameters $a, b, \cdots, c$), and the one-rowed matrix $(a, b, \cdots, c)$ is called the *characteristic* in each case; $x, y, \cdots, z$ are the arguments of the functions.

For the characteristics

$$(1, 0, 0, \cdots, 0),\quad (0, 1, 0, \cdots, 0), \cdots,\quad (0, 0, 0, \cdots, 1),$$

Λ_n is equal to $X_n, Y_n, \cdots, Z_n$ respectively, so that the common Bernoullian

functions are included in the special case $k = 1$. For the characteristics

$$(a, b, 0, \cdots, 0), \quad (a, 0, c, \cdots, 0), \cdots$$

we have $\frac{1}{2}k(k-1)$ functions of order 2, of which a typical one of rank n is

$$\sum \frac{n!}{\xi!\,\eta!}\, a^{\xi} b^{\eta} X_{\xi} Y_{\eta}, \quad 0 \leqq \xi,\ \eta; \quad \xi + \eta = n,$$

so that the functions of Krause and Appell * are given by the special case $k = 2$ with the further specialization that x, y be integers $\geqq 0$. If x, y, $\cdots$, z are integers, the explicit form of Λ_n is obtained at once from (2) rewritten as

$$e^{\Lambda\theta} = \sum_{\xi=1}^{x} \sum_{\eta=1}^{y} \cdots \sum_{\zeta=1}^{z} e^{(\xi a + \eta b + \cdots + \zeta c)\theta};$$

so that

$$\Lambda_n = \sum_{\xi=1}^{x} \sum_{\eta=1}^{y} \cdots \sum_{\zeta=1}^{z} (\xi a + \eta b + \cdots + \zeta c)^n.$$

The like is not true, of course, if x, y, $\cdots$, z are not integers > 0.

From (4), (6) we have

$$\Phi_n^{(a, b, \ldots, c)}(1, 1, \cdots, 1) = \Omega_n^{(a, b, \ldots, c)}; \tag{7}$$

and from (3), (5),

$$\Psi_n^{(a, b, \ldots, c)}(1, 1, \cdots, 1) = \Theta_n^{(a, b, \ldots, c)}. \tag{8}$$

For the respective characteristics

$$(a, 1, 1, \cdots, 1), \quad (1, b, 1, \cdots, 1), \cdots, \quad (1, 1, 1, \cdots, c),$$

Ω_n becomes $a^n A_n$, $b^n B_n$, $\cdots$, $c^n C_n$, where as always each of A, B, $\cdots$, C is an umbra of the Bernoulli numbers.

6. The relations between the numbers and functions defined in § 5 for various ranks are obtained by the following entirely general symbolic method. Let θ denote a parametric ordinary, and δ, α, β, $\cdots$, γ umbrae such that

$$\delta = \alpha + \beta + \cdots + \gamma, \tag{9}$$

or what is the same by the definition of umbral addition,

$$\delta^n = (\alpha + \beta + \cdots + \gamma)^n \quad (n = 0, 1, 2, \cdots). \tag{10}$$

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* *Ber. Ges. Leipzig*, 55 (1903), pp. 39-62; *Archiv Math. Phys.* (3) 4 (1902), pp. 292-5.

Let the c_n $(n = 0, 1, 2, \cdots)$ denote ordinaries, and write

$$f(\theta) \equiv \sum_{n=0}^{r} c_n \theta^n, \tag{11}$$

where the upper limit r of the summation may be finite or infinity. If the latter, we shall assume for brevity that all infinite series occurring henceforth are absolutely convergent for some $|\theta| \neq 0$. This assumption is not a necessity.* From (11) we have by means of (10)

$$f(\theta+\delta) = \sum_{n=0}^{r} c_n \Big[\sum_{s=0}^{n} \binom{n}{s} \theta^{n-s} (\alpha+\beta+\cdots+\gamma)^s \Big],$$

$$= \sum_{n=0}^{r} c_n (\theta+\alpha+\beta+\cdots+\gamma)^n,$$

and therefore

$$f(\theta+\delta) = f(\theta+\alpha+\beta+\cdots+\gamma); \tag{12}$$

that is, from (9) we infer (12), where $f(\theta)$ is arbitrary except that is possesses a power series expansion (which may terminate) in θ.

To obtain umbral identities such as (9), we observe that, if $P(\theta)$, $Q(\theta)$, $\cdots$, $T(\theta)$ are ordinary functions of θ whose power series expansions in θ are given by the umbral equivalents $e^{\delta\theta}$, $e^{\alpha\theta}$, $e^{\beta\theta}$, $\cdots$, $e^{\gamma\theta}$ respectively, so that we have the symbolic equations

$$P(\theta) = e^{\delta\theta}, \quad Q(\theta) = e^{\alpha\theta}, \quad R(\theta) = e^{\beta\theta}, \cdots, \quad T(\theta) = e^{\gamma\theta},$$

and if moreover we have the ordinary identity

$$P(\theta) = Q(\theta)\, R(\theta) \cdots T(\theta), \tag{13}$$

then

$$e^{\delta\theta} = e^{(\alpha+\beta+\ldots+\gamma)\theta}, \tag{14}$$

and hence we have (9). Hence the deduction of additive umbral identities such as (9) is reduced to that of multiplicative ordinary identities.

Consider next the frequently occurring modification of (13):

$$K\theta^r P(\theta) = Q(\theta)\, R(\theta) \cdots T(\theta), \tag{15}$$

where K is an ordinary constant (independent of θ) and r an integer > 0. Then, instead of (14) we have

$$K\theta^r e^{\delta\theta} = e^{(\alpha+\beta+\ldots+\gamma)\theta},$$

* In the paper cited, I have shown how the contrary case is to be interpreted.

which can be written symbolically,

$$K \quad \frac{\partial^r e^{\delta\theta}}{\partial \delta^r} = e^{(\alpha+\beta+\cdots+\gamma)\theta},$$

and hence

$$K \quad \frac{\partial^r \delta^n}{\partial \delta^r} = (\alpha+\beta+\cdots+\theta)^n;$$

whence, as in deriving (12), and eliminating the umbral derivative by means of

$$\frac{\partial^r}{\partial \delta^r}\ (\theta+\delta)^n = \frac{\partial^r}{\partial \theta^r}\ (\theta+\delta)^n,$$

we get, as the relation implied by (16),

$$(16) \qquad K \ \frac{\partial^r}{\partial \theta^r}\ f(\theta+\delta) = f(\theta+\alpha+\beta+\cdots+\gamma).$$

Hence by means of the simplest multiplicative identities between the ordinary functions defining umbrae we pass immediately to the generalized recurrences (12), (16). The process can evidently be applied as above to any functions δ_n, α_n, β_n, $\cdots$, γ_n $(n = 0, 1, 2, \cdots)$ which are coefficients in power series expansions.* The coefficients themselves may be, as above, functions of several variables.

7. By means of the processes in § 6 the relations between the functions defined in § 5 can be obtained with great ease. Change θ into $-\theta$ in (2) and compare the result with (2). Then, for $n = 0, 1, 2, \cdots$,

$$[a(x+1)+b(y+1)+\cdots+c(z+1)-\Lambda]^n = \Lambda^n,$$

and therefore

$$[-a(x+1)-b(y+1)-\cdots-c(z+1)+\Lambda]^n = (-\Lambda)^n,$$

from which, by (12), it follows that

$$(17) \quad \begin{aligned} &f(\theta-\Lambda+a(x+1)+b(y+1)+\cdots+c(z+1)) = f(\theta+\Lambda), \\ &f(\theta+\Lambda-a(x+1)-b(y+1)-\cdots-c(z+1)) = f(\theta-\Lambda). \end{aligned}$$

The special case in which Λ_n degenerates to the ordinary Bernoullian function X_n is obtained from (17) by taking $a = 1$ and deleting the terms in $b, \cdots, c$:

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* If, for example, $R(\theta) = \Sigma\rho_n\theta^n$ defines ρ_n, we first replace ρ_n by $n!\rho'_n$, so that

$$R(\theta) = \Sigma\rho'_n\ \theta^n/n! = e^{\rho'\theta},$$

and in the final results write $\rho'_n \equiv \rho_n/n!$

$$f(\theta \mp X \pm x \pm 1) = f(\theta \pm X), \tag{18}$$

the upper or the lower signs, as always, being taken throughout. For the functions of Krause and Appell only the a, b terms are retained from those in $a, b, \cdots, c$, and further, x, y are restricted to be integers. Likewise in all subsequent formulas concerning Λ or the functions Ψ, Φ.

In exactly the same way (3) gives

$$f(\theta \pm \Psi) = (-1)^k f(\theta \mp \Psi \pm ax \pm by \pm \cdots \pm cz), \tag{19}$$

and (6) gives

$$f(\theta \pm \Phi) = f(\theta \mp \Phi \mp ax \mp by \mp \cdots \mp cz). \tag{20}$$

Proceeding similarly with (4), (5) we get for the numbers Ω, Θ the following,

$$f(\theta \pm \Omega) = f(\theta \mp \Omega \mp a \mp b \mp \cdots \mp c), \tag{21}$$

which could have been written down from (20) by taking in the latter $(x, y, \cdots, z) = (1, 1, \cdots, 1)$; and

$$f(\theta \pm \Theta) = (-1)^k f(\theta \mp \Theta \pm a \pm b \pm \cdots \pm c), \tag{22}$$

which follows also from (19).

Corresponding to the relations between the common Bernoullian functions X_n and the Bernoulli numbers B_n for various ranks n, we have here the like for Λ_n, Ω_n, and the latter include the former as special cases. For (2), (4) with θ replaced by $-\theta$, (3) and (6) give immediately

$$(-1)^k ab \cdots c \; \frac{\partial^k}{\partial \theta^k} f(\theta \pm \Lambda) = f(\theta \pm \Psi \mp \Omega). \tag{23}$$

Further general relations for Λ are obtained from (2), (5),

$$f(\theta + \Lambda + \Theta - a - b - \cdots - c) = f(\theta + \Psi); \tag{24}$$

from (3), (5), and (2) with θ replaced by $-\theta$,

$$f(\theta \mp \Lambda \pm \Theta) = f(\theta \pm \Psi \mp ax \mp by \mp \cdots \mp cz); \tag{25}$$

from (2), (6),

$$f(\theta + \Lambda + \Phi) = xy \cdots zf(\theta + \Omega + a + b + \cdots + c), \tag{26}$$

and from (2), (6), the latter with θ replaced by $-\theta$,

$$(27)\qquad f(\theta+\Lambda-\Phi)$$
$$=xy\cdots zf(\theta+\Omega+a(1+x)+b(1+y)\ +\cdots+c(1+z)).$$

From (4), (5) we have

$$e^{(\Omega+\Theta)\theta}=(-1)^k ab\cdots c\theta^k,\ =(-1)^k ab\cdots ce^{\epsilon\theta},$$

where $\epsilon_k=k!$, $\epsilon_n=0$, $n\neq k$, and therefore

$$(28)\qquad f(\theta+\Omega+\Theta)=(-1)^k ab\cdots cf(\theta+\epsilon).$$

As an example of another type which, however, can be deduced from the foregoing by properly choosing f, let $\sum_2$ denote a summation with respect to all sets of 2 chosen from $ax, by, \cdots, cz$, a typical set of 2 being ax, by, and similarly for $\sum_r$, referring to all sets of r. Then from (2), and (4) with θ replaced by $-\theta$, we have

$$(-1)^k ab\cdots c\ \frac{\partial^k}{\partial\Lambda^k}\ e^{\Lambda\theta}=e^{-\Omega\theta}\ (1-e^{ax\theta})(1-e^{by\theta})\cdots(1-e^{cz\theta}),$$
$$=e^{-\Omega\theta}-\sum_1 e^{(-\Omega+ax)\theta}+\sum_2 e^{(-\Omega+ax+by)\theta}-\cdots+(-1)^k e^{(-\Omega+ax+\cdots+cz)\theta};$$

whence, as in deducing (12), (16), we get

$$(29)\qquad (-1)^k ab\cdots c\ \frac{\partial^k}{\partial\theta^k}\ f(\theta\pm\Lambda)$$
$$=f(\theta\mp\Omega)-\sum_1 f(\theta\mp\Omega\pm ax)+\sum_2 f(\theta\mp\Omega\pm ax\pm by)-\cdots.$$

The similar theorem for Ω is found from (4) rewritten in the form

$$(-1)^k ab\cdots c\theta^k=e^{\Omega\theta}\ (1-e^{a\theta})(1-e^{b\theta})\cdots(1-e^{c\theta}),$$

$$(30)\qquad ab\cdots c\ \frac{1}{k!}\ \frac{\partial^k}{\partial\theta^k}\ f(\theta)$$
$$=f(\theta+\Omega)-\sum_1 f(\theta+\Omega+a)+\sum_2 f(\theta+\Omega+a+b)-\cdots.$$

The power of the symbolic method is well seen from the derivation of general addition theorems. Let $(a', b', \cdots)$, $(x', y', \cdots)$ be the characteristic and arguments of Z', $(a'', b'', \cdots)$, $(x'', y'', \cdots)$ those of Z'', where Z', Z'' are not necessarily of the same order (same number of arguments $x', \cdots, x'', \cdots$), where Z denotes any one of Λ, Ψ, Φ. Then from (2), (3), (6) it follows that

$$e^{Z'\theta}\cdot e^{Z''\theta}=e^{Z\theta},$$

where Z is the function whose characteristic and arguments are $(a', b', \cdots, a'', b'', \cdots)$, $(x', y', \cdots, x'', y'', \cdots)$ respectively. Hence

$$f(\theta + Z) = f(\theta + Z' + Z''). \tag{31}$$

If in the foregoing we take $x' = y' = \cdots = x'' = y'' = \cdots = 1$, when Z is either Φ or Ψ we get

$$f(\theta + T) = f(\theta + T' + T''), \tag{32}$$

where T is either Ω or Θ.

As a last illustration of the power of this method let us find the partial derivatives of Λ, Ψ, Φ with respect to any of their arguments, say x. If Z_n is a function of x, and Z'_n its derivative with respect to x, it is readily seen that the first of the following implies the second,

$$F(x, \theta) = e^{Z\theta}, \quad \frac{\partial}{\partial x} F(x, \theta) = e^{Z'\theta}.$$

To apply this to Λ, Ψ, Φ, take the logarithmic derivatives with respect to x of (2), (3), (5) and proceed as before, getting

$$x \frac{\partial \Lambda}{\partial x} = (\Lambda + aAx + ax)^n, \tag{33}$$

$$x \frac{\partial \Psi_n}{\partial x} = (\Psi + aAx + ax)^n, \tag{34}$$

$$x \frac{\partial \Phi_n}{\partial x} = \Phi_n - (\Phi + aAx + ax)^n, \tag{35}$$

in which, we recall, A is an umbra of the Bernoulli numbers. Applied to (1), this gives for the common Bernoullian functions X,

$$\frac{\partial X_n}{\partial x} = (x - A)^n; \tag{36}$$

whence, integrating, we obtain

$$X_n = \frac{1}{n+1} (x - A)^{n+1}, \tag{37}$$

it being readily seen that the constant of integration is zero.

It may be mentioned in conclusion that the application of the symbolic method to Taylor's theorem for functions of k variables, together with the preceding relations involving functions f of the several umbrae, leads at once to interesting generalizations of several well known expansions due to Euler, Boole and others in the calculus of finite differences.

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The Construction of Certain Periodic Orbits of the Three Body Problem.

By H. E. Buchanan.

In a recent paper * the author proved the existence of certain periodic orbits of three finite masses near the LaGrangian circular solutions. The present paper is for the purpose of constructing those solutions and discussing the geometrical character of the orbits.

The differential equations which define the variations from the circular orbits are

$$\left.\begin{aligned}
&\frac{d^2x_1}{dt^2} - 2\omega(1+\delta)\frac{dy_1}{dt} - (1+\delta)^2\{(\omega^2+2A_1)x_1 + 2B_1x_3\} = (1+\delta)^2\epsilon \\
&\quad\left\{ x_1^2\frac{\partial^3U}{\partial x_1^3} + x_1x_3\frac{\partial^3U}{\partial x_1^2\partial x_3} + x_3^2\frac{\partial^3U}{\partial x_1\partial x_3^2} + \sum y_ky_j\frac{\partial^3U}{\partial x_1\partial y_k\partial y_j} \right. \\
&\qquad\left. + \sum z_kz_j\frac{\partial^3U}{\partial x_1\partial z_k\partial z_j} \right\} + \cdots, \\
&\frac{d^2x_3}{dt^2} - 2\omega(1+\delta)\frac{dy_3}{dt} - (1+\delta)^2\{(2B_1x_1 + (\omega^2+2A_3)x_3\} = (1+\delta)^2\epsilon \\
&\quad\left\{ x_1^2\frac{\partial^3U}{\partial x_3\partial x_1^2} + x_1x_3\frac{\partial^3U}{\partial x_3^2\partial x_1} + x_3^2\frac{\partial^3U}{\partial x_3^3} + \sum y_ky_j\frac{\partial^3U}{\partial x_3\partial y_k\partial y_j} \right. \\
&\qquad\left. + \sum z_kz_j\frac{\partial^3U}{\partial x_3\partial z_k\partial z_j} \right\} + \cdots, \\
&\frac{d^2y_1}{dt^2} + 2\omega(1+\delta)\frac{dx_1}{dt} - (1+\delta)^2\{(\omega^2-A_1)y_1 - B_1y_3\} = (1+\delta)^2\epsilon \\
&\quad\left\{ x_1y_1\frac{\partial^3U}{\partial y_1^2\partial x_1} + x_1y_3\frac{\partial^3U}{\partial y_1\partial y_3\partial x_1} + x_3y_1\frac{\partial^3U}{\partial y_1^2\partial x_3} + x_3y_3\frac{\partial^3U}{\partial y_1\partial y_3\partial x_3} \right\} + \cdots, \\
&\frac{d^2y_3}{dt^2} + 2\omega(1+\delta)\frac{dx_3}{dt} - (1+\delta)^2\{(\omega^2-A_3)y_3 - B_3y_1\} = (1+\delta)^2\epsilon \\
&\quad\left\{ x_1y_1\frac{\partial^3U}{\partial y_1\partial y_3\partial x_1} + x_1y_3\frac{\partial^3U}{\partial y_3^2\partial x_1} + x_3y_1\frac{\partial^3U}{\partial y_3\partial y_1\partial x_3} + x_3y_3\frac{\partial^3U}{\partial y_3^2\partial x_3} \right\} + \cdots, \\
&\frac{d^2z_1}{dt^2} + (1+\delta)^2(A_1z_1 + B_1z_3) = (1+\delta)^2\epsilon\sum x_kz_j\frac{\partial^3U}{\partial z_1\partial z_j\partial x_k} + \cdots, \\
&\frac{d^2z_3}{dt^2} + (1+\delta)^2(B_3z_1 + A_3z_3) + (1+\delta)^2\epsilon\sum x_kz_j\frac{\partial^3U}{\partial z_3\partial z_j\partial x_k} + \cdots,
\end{aligned}\right\} \quad (1)$$

* *American Journal of Mathematics*, Vol. XLV, Apr. 1923.

where the Σ denotes a sum of the form $Ay_1^2 + By_1y_3 + Cy_3^2$ and all the constants have been defined in the paper referred to.

Orbits with Period $2\pi/\nu$. We have demonstrated the existence of solutions of equations (1) which have the following properties:

(a) $$x_i = \sum_{j=0}^{\infty} x_{ij}\epsilon^j, \qquad y_i = \sum_{j=0}^{\infty} y_{ij}\epsilon^j, \quad z_i = \sum_{j=0}^{\infty} z_{ij}\epsilon^j, \quad (i = 1, 3),$$

$$\delta = \sum_{j=0}^{\infty} \delta_j\epsilon^j, \qquad a_i = \sum_{j=0}^{\infty} a_{ij}\epsilon^j, \quad \beta_i = \sum \beta_{ij}\epsilon^j$$

(b) Each x_{ij}, y_{ij}, z_{ij}, δ_j, a_{ij}, β_{ij} is uniquely determined,
(c) $x'_i = y_i = z_i = 0$ and $z'_i = c_i + y_i$ at $t = 0$,
(d) The series are convergent for $|\epsilon|$ sufficiently small,
(e) The whole solution and therefore each x_{ij}, y_{ij}, z_{ij}, is periodic with the period $2\pi/\nu$.

The coefficients of ϵ^0. The terms of the power series which are independent of ϵ are determined by a set of linear homogeneous differential equations easily found from (1) by making $\epsilon = 0$. Their solutions satisfying condition c are

$$\begin{aligned}
x_{i0} &= K_{i1}\cos\rho_1 t + K_{i2}\cos\omega t + K_{i3}\cosh\rho_2 t,\\
y_{i0} &= L_{i1}\sin\rho_1 t + L_{i2}\sin\omega t + L_{i3}\sinh\rho_2 t + L_{i4}t,\\
z_{i0} &= M_{i1}\sin\omega t + M_{i2}\sin\nu t.
\end{aligned}$$

Where K_{ij} and L_{3j} are uniquely determined in terms of L_{1j} by the equations

$$\begin{aligned}
&(\lambda^2 - \omega^2 - 2A_1)K_{1j} - 2B_1K_{3j} - 2\lambda\omega L_{1j} = 0,\\
&-2B_3K_{1j} + (\lambda^2 - \omega^2 - 2A_3)K_{3j} - 2\lambda\omega L_{3j} = 0,\\
&2\lambda\omega K_{1j} + (\lambda^2 - \omega^2 + A_1)L_{1j} + B_1L_{3j} = 0, \qquad j = 1, 2, 3.
\end{aligned}$$

Let the initial conditions be

$$x_i = a_i, \qquad y'_i = \beta_i, \qquad z'_i = c_i + \gamma_i \text{ at } t = 0.$$

The a_i, β_i are related to the L_{1j} by the equations.

$$\left.\begin{aligned}
a_1^{(0)} &= a_1L_{11} + a_3L_{12} + a_5L_{13},\\
a_3^{(0)} &= b_1L_{11} + b_3L_{12} + b_5L_{13},\\
\beta_1^{(0)} &= \rho_1\sqrt{-1}\,L_{11} + \omega\sqrt{-1}\,L_{12} + \rho_2L_{13} + L_{14},\\
\beta_3^{(0)} &= c_1\rho_1\sqrt{-1}\,L_{11} + c_3\omega\sqrt{-1}\,L_{12} + c_5\rho_2L_{13} + c_7L_{14},
\end{aligned}\right\} \quad (2)$$

Where $a_1^{(0)}$, $a_3^{(0)}$, $\beta_1^{(0)}$ and $\beta_3^{(0)}$ are the terms independent of ϵ in the power series which give a_1, a_3, β_1 and β_3 in terms of ϵ.

The periodicity condition (d) demands that

$$L_{11} = L_{12} = L_{13} = L_{14} = 0.$$

Hence

$$a_1^{(0)} = a_3^{(0)} = \beta_1^{(0)} = \beta_3^{(0)} = 0.$$

The M_{ij} are related to the initial constants by the equations

$$\left.\begin{aligned} c_1 + \gamma_1 &= \omega M_{11} + \nu M_{12}, \\ c_3 &= \gamma_3 - \omega \frac{\omega^2 - A_1}{B_1} M_{11} + \nu \ M_{12} \end{aligned}\right\} \quad (3)$$

The periodicity condition (d) also requires that $M_{11} = 0$, hence

$$c_3 + \gamma_3 = \frac{\nu^2 - A_1}{B_1} \ (c_1 + \gamma_1)$$

and

$$M_{12} = \frac{c_1 + \gamma_1}{\nu}$$

The terms of the series which are independent of ϵ are

$$x_{i0} = y_{i0} = 0,$$

$$z_{10} = \frac{c_1 + \gamma_1}{\nu} \sin \nu t,$$

$$z_{30} = \frac{\nu^2 - A_1}{B_1} \ \frac{c_1 + \gamma_1}{\nu} \sin \nu t.$$

The coefficients of ϵ.—These are determined by the following equations whose solutions must be subjected to the conditions (a), (b), (c), (d) and (e):

$$\left.\begin{aligned}
&\frac{d^2x_{11}}{dt^2} - 2\omega\frac{dy_{11}}{dt} - \{(\omega^2 + 2A_1)x_{11} + 2B_1x_{31}\} = z^2{}_{10}\frac{\partial^3 U}{\partial x_1 \partial z_1{}^2} \\
&\qquad + z_{10}z_{30}\frac{\partial^3 U}{\partial x_1 \partial z_1 \partial z_3} + z^2{}_{30}\frac{\partial^3 U}{\partial x_1 \partial z_3{}^2}, \\
&\frac{d^2x_{31}}{dt^2} - 2\omega\frac{dy_{31}}{dt} - \{2B_3x_{11} + (\omega^2 + 2A_3)x_{31}\} = z^2{}_{10}\frac{\partial^3 U}{\partial x_3 \partial z_1{}^2} \\
&\qquad + z_{10}z_{30}\frac{\partial^3 U}{\partial x_3 \partial z_1 \partial z_3} + z^2{}_{30}\frac{\partial^3 U}{\partial x_3 \partial z_3{}^2}, \\
&\frac{d^2y_{11}}{dt^2} + 2\omega\frac{dx_{11}}{dt} - (\omega^2 - A_1)y_{11} + B_1y_{31} = 0, \\
&\frac{d^2y_{31}}{dt^2} + 2\omega\frac{dx_{31}}{dt} + B_3y_{11} - (\omega^2 - A_3)y_{31} = 0, \\
&\frac{d^2z_{11}}{dt^2} + A_1z_{11} + B_1z_{31} = -2\delta_1\{A_1M_{12}\sin\nu t + B_1M_{32}\sin\nu t\}, \\
&\frac{d^2z_{31}}{dt^2} + B_3z_{11} + A_3z_{31} = -2\delta_1\{B_3M_{12}\sin\nu t + A_3M_{32}\sin\nu t\}.
\end{aligned}\right\} \quad (4)$$

The first four equations are independent of the last two. Their left members are the same as the left members of the equations which determined x_{i0} and y_{i0}. The right members can be thrown in the form $A_{10}{}^{(1)} + A_{12}{}^{(1)} \cos 2\nu t$, $A_{30}{}^{(1)} + A_{32}{}^{(1)} \cos 2\nu t$, 0, 0 where the A's all carry the arbitrary $c_1 + \gamma_1$ as a quadratic factor.

The solution of (53) consists of a complementary function plus a particular integral. The complementary function has the wrong period as in the article just preceding and hence must vanish.

The particular integrals are

$$x_{i1} = K_{i0}{}^{(1)} + K_{i2}{}^{(1)} \cos 2\nu t,\ y_{i1} = L_{i0}{}^{(1)} + L_{i2}{}^{(1)} \sin 2\nu t$$

where $K_{i0}{}^{(1)}$, $K_{i2}{}^{(1)}$, $L_{i0}{}^{(1)}$ and $L_{i2}{}^{(1)}$ are determined so as to satisfy the differential equations. Applying the initial conditions we find

$$\alpha_i{}^{(1)} = K_{i0}{}^{(1)} + K_{i2}{}^{(1)},\ \beta_i{}^{(1)} = 2\nu L_{i2}{}^{(1)} \qquad i = 1,\ 3.$$

In the last two equations of (4) the solution again consists of a complementary function plus a particular integral. But the right members have the same period as the complementary function. Therefore in order that the solution be periodic the coefficients of the right members must be zero. These coefficients are

$$-2\delta_1(A_1M_{12} + B_1M_{32}),\quad -2\delta_1(B_3M_{12} + A_3M_{32}).$$

From the relation $M_{32} = \dfrac{\nu^2 - A_1}{B_1} M_{12}$ it is clear that the second factor cannot vanish hence $\delta_1 = 0$. Since the right members vanish the particular integrals are zero. The initial conditions by (c) are $z_{i1} = z'_{i1} = 0$ at $t = 0$. Therefore the complementary function is zero and the solutions satisfying the conditions which have been imposed are

$$z_{i1} = 0.$$

The coefficients of ϵ^2. These coefficients are determined by the equations

$$\left.\begin{aligned}
&\frac{d^2x_{13}}{dt^2} - 2\omega\frac{dy_{12}}{dt} - \{(\omega^2 + 2A_1)x_{12} + 2B_1x_{32}\} = 0,\\
&\frac{d^2x_{32}}{dt^2} - 2\omega\frac{dy_{32}}{dt} - \{2B_3x_{12} + (\omega^2 + 2A_3)x_{32}\} = 0,\\
&\frac{d^2y_{12}}{dt^2} + 2\omega\frac{dx_{12}}{dt} - (\omega^2 - A_1)y_{12} + B_1y_{32} = 0,\\
&\frac{d^2y_{32}}{dt^2} + 2\omega\frac{dx_{32}}{dt} + B_3y_{12} - (\omega^2 - A_3)y_{32} = 0,\\
&\frac{d^2z_{12}}{dt^2} + A_1z_{12} + B_1z_{32}\\
&\qquad = -2\delta_2(A_1M_{12} + B_1M_{32})\sin\nu t + c_{11}^{(2)}\sin\nu t + c_{13}^{(2)}\sin 3\nu t,\\
&\frac{d^2z_{32}}{dt^2} + B_3z_{12} + A_3z_{32}\\
&\qquad = -2\delta_2(B_3M_{12} + A_3M_{32})\sin\nu t + c_{32}^{(2)}\sin\nu t + c_{33}^{(2)}\sin 3\nu t.
\end{aligned}\right\} (5)$$

The solution of the first four equations consists simply of the complementary function. It has the wrong period and must vanish thus determining $\alpha_1^{(2)}$, $\alpha_3^{(2)}$, $\beta_1^{(2)}$ and $\beta_3^{(2)}$ all of which are zero. Hence

$$x_{i2} = y_{i2} = 0,$$

are the solutions satisfying the conditions.

In the last two equations the coefficients of $\sin \nu t$ must vanish giving

$$\begin{aligned}
-2\delta_1(A_1M_{12} + B_1M_{32}) + c_{11}^{(2)} &= 0,\\
-2\delta_1(B_3M_{12} + A_3M_{32}) + c_{31}^{(2)} &= 0.
\end{aligned}$$

Since $M_{32} = \dfrac{\nu^2 - A_1}{B_1} M_{12}$ these give

$$\delta_2 = \frac{c_{11}^{(2)}}{2\nu M_{12}}, \text{ and } \delta_2 = \frac{B_1c_{31}^{(2)}}{2\nu^2(\nu^2 - A_1)M_{12}}.$$

By direct computation one can verify that

$$c_{31}^{(2)} = \frac{\nu^2 - A_1}{B_1}\ c_{11}^{(2)}.$$

Hence the two values of δ_2 are the same. The right members of the last two equations of (5) will contain only sines of odd multiples of t. Therefore the complete solution is

$$z_{i2} = M_{i1} \sin \omega t + M_{i2} \sin \nu t + K_i^{(2)} \sin 3\nu t,$$

where the $K_i^{(2)}$ have been determined by the usual method for finding the particular integral. By the initial conditions

$$\omega M_{11} + \nu M_{12} + 3\nu K_1^{(2)} = 0,$$

$$\omega\ \frac{\omega^2 - A_1}{B_1}\ M_{11} + \nu\ \frac{\nu^2 - A_1}{B_1}\ M_{12} + 3\nu K_3^{(2)} = \gamma_3^{(2)}.$$

Hence setting $M_{11} = 0$ as the periodicity conditions require we get

$$\gamma_3^{(2)} = 3\nu K_3^{(2)} - 3\nu K_1^{(2)}\ \frac{\nu^2 - A_1}{B_1}\ .$$

A direct determination of the K's shows that this value of $\gamma_3^{(2)}$ is not zero.

The Normal Form for the Z-Equations. In each step of the construction there will appear two determinations of δ_i. In order to show that these two values are always the same we make a transformation of the form

$$\left.\begin{aligned} z_1 &= v_1 + v_2 + v_3 + v_4, \\ z'_1 &= \omega\sqrt{-1}(v_1 - v_2) + \nu\sqrt{-1}(v_3 - v_4), \\ z_3 &= d_1(v_1 + v_2) + d_2(v_3 + v_4), \\ z'_3 &= \omega\sqrt{-1}\, d_1(v_1 - v_2) + \nu\sqrt{-1}\, d_2(v_3 - v_4). \end{aligned}\right\} \tag{6}$$

In the normal variables the Z-equations of motion become

$$\left.\begin{aligned} v'_1 &= (1+\delta)\omega\sqrt{-1}\, v_1 + (1+\delta)\epsilon\{C_1R_1 + C_3R_3\}, \\ v'_2 &= -(1+\delta)\omega\sqrt{-1}\, v_1 + (1+\delta)\epsilon\{-C_1R_1 - C_3R_3\}, \\ v'_3 &= (1+\delta)\nu\sqrt{-1}\, v_3 + (1+\delta)\epsilon\{-C_1R_1 + C_3R_3\}, \\ v'_4 &= -(1+\delta)\nu\sqrt{-1}\, v_4 + (1+\delta)\epsilon\{C_1R_1 - C_3R_3\}, \end{aligned}\right\} \tag{7}$$

Where R_1 and R_3 are the right members of the original Z-equations. It is not necessary to change them into v's.

If we had made the previous part of the construction in the new variables it is clear that the coefficient of $e^{\nu\sqrt{-1}\, t}$ is the same in the v_3 and v_4 equations. Hence the two determinations of δ_2 are obviously the same.

The general term. Suppose that the

$$x_{i0}, \cdots x_{in}, \quad y_{i0}, \cdots y_{in}, \; z_{i0}, \cdots z_{in}, \quad \delta_1, \cdots \delta_{n-1},$$

have been determined and have the following properties:

(1) $x_{i2k} = y_{i2k} = z_{i2k+1} = \delta_{2k+1} = 0, \; \delta_{2k} \neq 0.$

(2) x_{i2k+1} are cosines of even multiples of νt, the highest being $2(k+1)$.

(3) y_{i2k+1} are sines of even multiples of νt, the highest being $2(k+1)$.

(4) z_{i2k} are sines of odd multiples of νt, the highest being $2k+1$.

It is possible to make an induction showing that these properties persist. This was done by Moulton in his Oscillating Satellite paper, for the case of an infinitesimal body. The details are so closely related to his work that it is not necessary to repeat it.

The character of the orbits. In the generating solutions $x_i = y_i = 0$, $z_i = M_i \sin \nu t$ there are apparently two constants M_1 and M_3 but $M_3 = \frac{\nu^2 - A_1}{B_1} M_1$ so that there is but one independent constant. The forms of the solutions show that $c_1 + \gamma_1$ enters as a factor of each x_{in}, y_{in} and z_{in} but to a power one greater than ϵ. Therefore, if we replace $(c_1 + \gamma_1)\epsilon$ by ϵ we get

$$x_i = (c_1 + \gamma_1) \sum_{j=0}^{\infty} x_{ij}\epsilon^j, \quad y_i = (c_1 + \gamma_1) \sum_{j=0}^{\infty} y_{ij}\epsilon^j, \quad z_i = (c_1 + \gamma_1) \sum_{j=0}^{\infty} z_{ij}\epsilon^j.$$

The coordinates in the orbits are given by

$$x'_i = \sum_{j=0}^{\infty} x_{ij}\epsilon^j, \quad y'_i = \sum_{j=0}^{\infty} y_{ij}\epsilon^j, \quad z'_i = \sum_{j=0}^{\infty} z_{ij}\epsilon^j,$$

where the primes denote the rectangular coordinates in rotating space. The exact forms of x'_i, y'_i, z'_i as far as the second power of ϵ' are

$$\left.\begin{aligned} x'_i &= \{K_{i0}^{(1)} + K_{i2}^{(1)} \cos \frac{2\nu t}{1+\delta}\}\epsilon'^2 + \cdots, \\ y'_i &= L_{i2}^{(1)} \sin \frac{2\nu t}{1+\delta}\, \epsilon'^2 + \cdots, \\ z'_i &= M_{i1}^{(1)} \sin \frac{\nu t}{1+\delta}\, \epsilon' + \cdots, \end{aligned}\right\} \tag{8}$$

The equations of the projections of the curves on the plane is found by eliminating t between the first two equations. The result is

$$\left\{\frac{x'_i - K_{i0}^{(1)}\epsilon'^2}{K_{i2}^{(1)}\epsilon'^2}\right\}^2 + \left\{\frac{y'_i}{L_{i2}^{(1)}\epsilon'^2}\right\}^2 = 1$$

which are the equations of ellipses whose centers are on the x-axis.

The projections on the x', z' are the parabolas

$$x'_i = \{K_{i0}^{(1)} + K_{i2}^{(1)}(1 - \frac{2z'^2}{(M_{i1}^{(1)})^2\epsilon'^2})\}\epsilon'^2.$$

The projections of the y', z' plane are

$$y'_i = 2L_{i2}^{(1)} \frac{z'_i}{M_{i1}^{(1)}\epsilon}\sqrt{1 - \frac{z_i'^2}{(M_{i1}^{(1)}\epsilon)^2}}$$

The orbits are therefore curves shaped like the figure 8. It can be proved from equations that the parabolas are opening towards each other and that the values of y_i for small values of t are negative. The orbits are therefore retrograde. These projections show their general shape with reference to the rotating axes. The orbit of the middle body is always determined by the center of gravity equations.

Orbits with the period $2\pi/\omega$. The properties of the solutions which were established in the existence proof are:

(a) each x_{ij}, y_{ij}, z_{ij}, δ_j is uniquely determined.

(b) $x'_i = y_i = z_i = 0$ and $z'_i = c_i + \gamma_i$ at $t = 0$.

(c) The series are convergent for $0 < |\epsilon| < \epsilon_0$, $0 < t < T$.

(d) The whole solution and therefore each x_{ij}, y_{ij}, z_{ij} is periodic with period $2\pi/\omega$.

Applying these properties we find the coefficient of ϵ^0 to be

$$x_{i0} = K_{i1}^{(0)} \cos \omega t, \quad y_{i0} = L_{i1}^{(0)} \sin \omega t, \quad z_{i0} = M_{i1}^{(0)} \sin \omega t$$

since by the condition (d) the coefficients of $\cos \rho_1 t$, $\cosh \rho_2 t$, $\cos \nu t$ and of t must vanish. That is, we have

$$L_{i1} = L_{i3} = L_{i4} = M_{i3} = 0 \ .$$

These equations are no longer homogeneous as in case 1 for the initial conditions are

$$x_{i0} = a_i + a_i^{(0)}, \quad y'_{i0} = b_i + \beta_i^{(0)}, \quad z'_{i0} = c_i + \gamma_i^{(0)}.$$

The first three equations have a determinant different from zero. Therefore

it is possible to solve for three of the initial constants in terms of the fourth, which was taken arbitrarly, and without loss of generality can be made equal to zero. The equation $M_{13} = 0$ gives γ_3 uniquely in terms of γ_1 which is so far undetermined. It will be necessary to determine γ_1 along with δ so that the solutions at the successive steps shall be periodic.

The coefficients of ϵ are determined by the differential equations.

$$\left.\begin{aligned}
&x''_{i1} - 2\omega y'_{i1} - (\omega^2 + 2A_i)x_{i1} - 2B_i x_{k1} \\
&\quad = 2\delta_1\{(\omega^2 + 2A_i)x_{i0} + 2B_i x_{k0}\} + A_{i0}^{(1)} + A_{i2}^{(1)} \cos 2\omega t, \\
&y''_{i1} + 2\omega y'_{i1} - (\omega^2 - A_i)y_{i1} + B_i y_{k1} \\
&\quad = 2\delta_1\{\omega^2 - A_i)y_{i0} - B_i y_{k0}\} + B_{i0}^{(1)} + B_{i2}^{(1)} \sin 2\omega t, \\
&z''_{i1} + A_i z_{i1} + B_i z_{k1} \\
&\quad = -2\delta_1(A_i z_{i0} + B_i z_{k0}) - \gamma_1^{(1)}(A_i z_{i0}^{(1)} + B_i z_{k0}^{(1)}) \\
&\qquad + C_{i0}^{(1)} + C_{i2}^{(1)} \sin 2\omega t, \quad (i = 1, 3, k = 3, 1).
\end{aligned}\right\} \tag{9}$$

The superscript notation here means that the part of z_{i0} going with the first power of ϵ in γ_1 is taken. In these six equations the first terms of the right members have the same period as the solutions of the left members set equal to zero. In order that the solutions shall be periodic we must have

$$\gamma_1^{(1)} = \delta_1 = 0.$$

Then the solutions become

$$\left.\begin{aligned}
x_{i1} &= A_{i0}^{(1)} + A_{i1}^{(1)} \cos \omega t + A_{i2}^{(1)} \cos 2\omega t, \\
y_{i1} &= B_{i0}^{(1)} + B_{i1}^{(1)} \sin \omega t + B_{i2}^{(1)} \sin 2\omega t, \\
z_{i1} &= C_{i0}^{(1)} + C_{i1}^{(1)} \sin \omega t + C_{i2}^{(1)} \sin 2\omega t,
\end{aligned}\right\} \tag{10}$$

where the $A_{ij}^{(1)}$, $B_{ij}^{(1)}$, $C_{ij}^{(1)}$ are determined so as to satisfy the differential equations.

In the computation of the coefficients of ϵ^2 the equations which determine $\gamma_1^{(2)}$ and δ_2 are no longer homogeneous as above. Further, there are six linear equations which these two quantities must satisfy in order that the solutions shall be periodic. In order to show that these six equations reduce to two independent ones it is convenient to transform to the normal variables for the remainder of the construction.

The equations of motion become in the new variables.*

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* See equations (22) *Am. Journal of Math.*, Apr. 1923, p. 101.

$$
\left.\begin{aligned}
u'_1 &= (1+\delta)\phi\sqrt{-1}\,u_1 + (1+\delta)A_{11}\epsilon\{\textstyle\sum x_k y_j \dfrac{\partial^3 U}{\partial y_1 \partial y_j \partial x_k} + \epsilon(\ )\cdots\} \\
&\qquad + (1+\delta)A_{12}\epsilon\{\textstyle\sum x_k y_j \dfrac{\partial^3 U}{\partial y_3 \partial y_j \partial x_k} + \epsilon(\ )\cdots\} \\
&+ (1+\delta)A_{13}\epsilon\{\textstyle\sum x_k x_j \dfrac{\partial^3 U}{\partial x_1 \partial x_k \partial x_j} + \epsilon(\ )\cdots \sum y_k y_j \dfrac{\partial^3 U}{\partial x_1 \partial y_k \partial y_j} + \epsilon(\ )\cdots \\
&+ \textstyle\sum z_k z_j \dfrac{\partial^3 U}{\partial x_1 \partial z_k \partial z_j} + \epsilon(\ )\cdots\} + (1+\delta)A_{14}\epsilon\{\textstyle\sum x_k x_j \dfrac{\partial^3 U}{\partial x_3 \partial x_k \partial x_j} + \epsilon(\ )\cdots \\
&+ \textstyle\sum y_k y_j \dfrac{\partial^3 U}{\partial x_3 \partial y_k \partial y_j} + \epsilon(\ )\cdots + \sum z_k z_j \dfrac{\partial^3 U}{\partial x_3 \partial z_k \partial z_j} + \epsilon(\ )\cdots\}, \\
u'_2 &= -(1+\delta)\phi_1\sqrt{-1}\,u_2 + (1+\delta)\epsilon \\
&\qquad \{-A_{11}(\ ) - A_{12}(\ ) - A_{13}(\ ) - A_{14}(\ )\}, \\
u'_3 &= (1+\delta)\omega\sqrt{-1}\,u_3 + (1+\delta)\epsilon \\
&\qquad \{B_{11}(\ ) + B_{12}(\ ) + B_{13}(\ ) + B_{14}(\ )\}, \\
u'_4 &= -(1+\delta)\omega\sqrt{-1}\,u_4 + (1+\delta)\epsilon \\
&\qquad \{-B_{11}(\ ) - B_{12}(\ ) - B_{13}(\ ) - B_{14}(\ )\}, \\
u'_5 &= (1+\delta)\phi_2 u_5 + (1+\delta)\epsilon \\
&\qquad \{C_{11}(\ ) + C_{12}(\ ) + C_{13}(\ ) + C_{14}(\ )\}, \\
u'_6 &= -(1+\delta)\phi_2 u_6 + (1+\delta)\epsilon \\
&\qquad \{-C_{11}(\ ) - C_{12}(\ ) - C_{13}(\ ) - C_{14}(\ )\}, \\
u'_7 &= (1+\ )\delta\epsilon\ \{D_{11}(\ ) + D_{12}(\ )\ \}, \\
u_8' &= u_7 + (1+\delta)\epsilon \\
&\qquad \{D_{11}(\ ) + D_{12}(\ ) - D_{13}(\ ) - D_{14}(\ )\}, \\
z_1'' &+ (1+\delta)^2(A_1 z_1 + B_1 z_3) = (1+\delta)^2\epsilon \textstyle\sum x_k z_j \dfrac{\partial^3 U}{\partial z_1 \partial x_k \partial z_j} \cdots, \\
z_3'' &+ (1+\delta)^2(B_3 z_1 + A_3 z_3) = (1+\delta)^2\epsilon \textstyle\sum x_k z_j \dfrac{\partial^3 U}{\partial z_3 \partial x_k \partial z_j} \cdots,
\end{aligned}\right\} \quad (11)
$$

where the parentheses denote the coefficients of A_{11}, A_{12}, A_{13}, A_{14} in the first equation.

For the sake of clearness we will compute again the coefficients of ϵ^0 and ϵ. Let the initial conditions be

$$
u_i = a_i, \quad (i = 1, 2, 5, 6, 7, 8), \qquad u_3 = a + a_3, \qquad u_4 = a + a_4,
$$
$$
z'_i = c_i + \gamma_i, \qquad c_i = \omega M_i.
$$

Therefore the coefficients of ϵ^0 are

$$u_{30} = (a + a_3)e^{\omega\sqrt{-1}\,t}, \quad u_{40} = (a + a_4)e^{\omega\sqrt{-1}\,t},$$
$$z_{i0} = M_{i0} \sin \omega t. \qquad u_{i0} = 0, \ (i = 1, 2, 5, 6, 7, 8),$$

By the orthogonality conditions $a_3 = a_4$ and therefore

$$\left.\begin{aligned} x_{10} &= 2a_1(a + a) \cos \omega t, \quad x_{30} = 2b_1(a + a) \cos \omega t, \\ y_{10} &= 2(a + a) \sin \omega t, \quad y_{30} = 2c_1(a + a \sin \omega t, \\ z_{i0} &= M_{i0} \sin \omega t. \end{aligned}\right\} \tag{12}$$

The coefficients of ϵ are determined by a set of equations like (11) in which all the terms containing ϵ to a higher power than the first are omitted and the values of x_{i0}, y_{i0}, z_{i0} are substituted from (12). It is easily seen that the periodicity conditions are satisfied by $\delta_1 = \gamma_1^{(1)} = 0$. The solutions are

$$\left.\begin{aligned} u_{11} &= a_1^{(1)} + a_2^{(1)} \sin 2\omega t + a_3^{(1)} \cos 2\omega t, \\ u_{21} &= a_1^{(1)} - a_2^{(1)} \sin 2\omega t + a_3^{(1)} \cos 2\omega t, \\ u_{31} &= b_0^{(1)} e^{\omega\sqrt{-1}\,t} + b_1^{(1)} + b_2^{(1)} \sin 2\omega t + b_3^{(1)} \cos 2\omega t, \\ u_{41} &= b_0^{(1)} e^{-\omega\sqrt{-1}\,t} + b_1^{(1)} - b_2^{(1)} \sin 2\omega t + b_3^{(1)} \cos 2\omega t, \\ u_{51} &= c_1^{(1)} + c_2^{(1)} \sin 2\omega t + c_3^{(1)} \cos 2\omega t, \\ u_{61} &= c_1^{(1)} - c_2^{(1)} \sin 2\omega t + c_3^{(1)} \cos 2\omega t, \\ u_{71} &= d_1^{(1)} + d_2^{(1)} \sin 2\omega t + d_3^{(1)} \cos 2\omega t, \\ u_{81} &= d_1^{(1)} - d_2^{(1)} \sin 2\omega t + d_3^{(1)} \cos 2\omega t, \\ z_{i1} &= M_{i1}^{(1)} \sin \omega t + c_{i1}^{(1)} + c_{i2}^{(1)} \sin \omega t. \end{aligned}\right\} \tag{13}$$

The two arbitrary constants of integration, $b_0^{(1)}$ and $b^{(1)}$ are determined by the orthogonality conditions so that $b_0^{(1)} = b^{(1)} = b/2$ where b is a known quantity. Therefore in the x_i, y_i, z_i variables we have

$$x_{i1} = A_{i0}^{(1)} + A_{i1}^{(1)} \cos \omega t + A_{i2}^{(1)} \cos 2\omega t,$$
$$y_{i1} = B_{i0}^{(1)} + B_{i1}^{(1)} \sin \omega t + B_{i2}^{(1)} \sin 2\omega t,$$
$$z_{i1} = C_{i0}^{(1)} + C_{i1}^{(1)} \sin \omega t + C_{i2}^{(1)} \sin 2\omega t.$$

The character of the differential equations together with the initial conditions shows that x_i will always be a cosine series, y_i a sine series, z_i a sine series, provided $\gamma_1^{(n)}\delta_n$ can be determined so that the solutions are periodic. We will prove that there are just enough independent equations to determine $\gamma_1^{(n)}$ and δ_n uniquely. The equations defining the coefficients of ϵ^n are

$$
\left.\begin{aligned}
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&u'_{3n} = \omega\sqrt{-1}\,u_{3n} + \delta_n\omega\sqrt{-1}\,u_{30} + A_{30}^{(n)} + A_{31}^{(n)}\sin\omega t\\
&\qquad + A_{32}^{(n)}\sin 2\omega t + \cdots + b_{31}^{(n)}\cos\omega t + b_{32}^{(n)}\cos 2\omega t + \cdots,\\
&u'_{4n} = -\omega\sqrt{-1}\,u_{4n} + \delta_n\omega\sqrt{-1}\,u_{30} + A_{30}^{(n)} + A_{31}^{(n)}\sin\omega t\\
&\qquad + A_{32}^{(n)}\sin 2\omega t + \cdots - b_{31}^{(n)}\cos\omega t - b_{32}^{(n)}\cos 2\omega t + \cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&z''_{4n} = -A_1 z_{1n} - B_1 z_{3n} - 2\delta_n(A_1 z_{10} + B_1 z_{30}) - \gamma_1^{(n)}(A_1 z_{10}^{(n)} + B_1 z_{30}^{(n)})\\
&\qquad + c_{11}^{(n)} + c_{12}^{(n)}\sin\omega t + c_{13}^{(n)}\sin 2\omega t + \cdots\\
&z''_{3n} = -B_3 z_n - A_3 z_{3n} - 2\delta_n(B_3 z_{10} + A_3 z_{30}) - \gamma_1^{(n)}(B_3 z_{10}^{(n)} + A_3 z_{30}^{(n)}\\
&\qquad + c_{31}^{(n)} + c_{32}^{(n)}\sin\omega t + c_{33}^{(n)}\sin 2\omega t + \cdots
\end{aligned}\right\} \tag{14}
$$

where the equations omitted are already in such a form that the periodicity conditions are satisfied. The condition that the first equation of (14) shall have a periodic solution is

$$
\omega\sqrt{-1}\,(a+\alpha)\delta_n + \frac{a_{31}^{(n)}}{2\sqrt{-1}} + \frac{b_{31}^{(n)}}{2} = 0
$$

since the coefficient of $e^{\omega\sqrt{-1}\,t}$ must vanish. In the second equation the coefficient of $e^{-\omega\sqrt{-1}\,t}$ must vanish, which gives exactly the same equation as the above. This equation uniquely determines δ_n.

In order that the last two equations shall give periodic solutions we must have

$$
\begin{aligned}
2\delta_n(A_1M_{10} + B_1M_{30}) + \gamma_1^{(n)}(A_1M_{10} + B_1M_{30}) &= \phi_{12}^{(n)},\\
2\delta_n(B_3M_{10} + A_3M_{30}) + \gamma_1^{(n)}(B_3M_{10} + A_3M_{30}) &= \phi_{32}^{(n)}.
\end{aligned}
$$

These two equations can be shown to be identical by reducing the z_i equations to the normal form as in the last article. Since δ_n is already known, $\gamma_1^{(n)}$ is uniquely determined by either of these equations.

In the generating solutions there are two independent arbitrary numbers L_1, M_1. It is no restriction on their independence to put $M_1 = CL_1$. Then in the solutions constructed above, L_1 enters as a factor of each x_{ij}, y_{ij}, z_{ij} but to a power one greater than ϵ. Therefore in the expressions for the coordinates, L_1 can be absorbed into ϵ. The C is left arbitrary.

We have investigated the orbits with the period $2\pi/\omega$ under general initial conditions, and under orthogonal initial conditions. Let us now consider the possibility of orbits with period $2\pi/\omega$ such that the coordinates are in the following proportion

$$
\frac{x_1}{x_3} = \frac{y_1}{y_3} = \frac{z_1}{z_3} = \text{a constant.}
$$

If we make this substitution in equations (2), they reduce to the two-body problem and are completely integrable.*

When the above proportion is satisfied the bodies all move in ellipses and remain in a straight line. The plane of motion may have any inclination whatever, which accounts for the arbitrary C left in the construction above.

Orbits with Period $2\pi/\rho_1$. We have already proved that these orbits lie wholly in the xy-plane. The method of constructing them is very similar to that of case 2 with the z-equations omitted. It is again necessary to transform to the normal form in order to show that the two determinations of δ_n are identical.

The geometrical character of the orbits is described by the following statements:

(1) The orbits are symmetrical with respect to the x-axis geometrically and the time.

(2) Omitting powers of ϵ beyond the first, the orbits are ellipses with centers on the x-axis.

Orbits in which ρ_1 *and* ν *are commensurable.* These are of the same character as the orbits with period $2\pi/\omega$. The same difficulties of construction arise and may be handled in a similar fashion. The orbits are symmetrical with respect to the x-axis geometrically and in the time. In case (2) we showed that the orbits lie wholly in one plane with an arbitrary inclination but in this case there does not seem to be such a simple interpretation of the constants. The projections of the orbits on the xy-plane; neglecting powers of ϵ beyond the first, are ellipses with centers on the x-axis. The period is the least common multiple of $2\pi/\rho_1$ and $2\pi/\nu$.

* Lagrange, *Collected Works*, VI, pp. 230-270.

Correction.

On page 223 of the current volume of this JOURNAL, the first parametric equation for curves (9) and (10) should read

$$x_0 = t^7 + at.$$

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